Introduction to the Quantum Hall Effect and Topological Phases

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Ecole du GDR “Physique Mésoscopique, Cargèse, November 2016

Historical Introduction

What is the common point between

- graphene,
- quantum Hall effects
- and topological insulators?

... and what is it?

The 1920ies: Band Theory

- quantum treatment of (non-interacting) electrons in a periodic lattice
- bands = energy of the electrons as a function of a quasi-momentum

1950-70: Many-Body Theory

- Physical System described by a local order parameter
  (a) $\Delta_{\mathbf{k}} = \langle \psi^{\dagger}_{-\mathbf{k},\uparrow} \psi^{\dagger}_{\mathbf{k},\downarrow} \rangle$ (superconductivity)
  (b) $M^{\mu}(\mathbf{r}) = \sum_{\tau,\tau'} \langle \psi^{\dagger}_{\tau}(\mathbf{r}) \sigma^{\mu}_{\tau,\tau'} \psi_{\tau'}(\mathbf{r}) \rangle$ (ferromagnetism)
- Ginzburg-Landau theory of second-order phase transitions (1957)

$$\begin{array}{ccc} \Delta = 0 & & \Delta \neq 0 \\ \text{(disordered)} & \leftrightarrow & \text{(ordered)} \end{array}$$

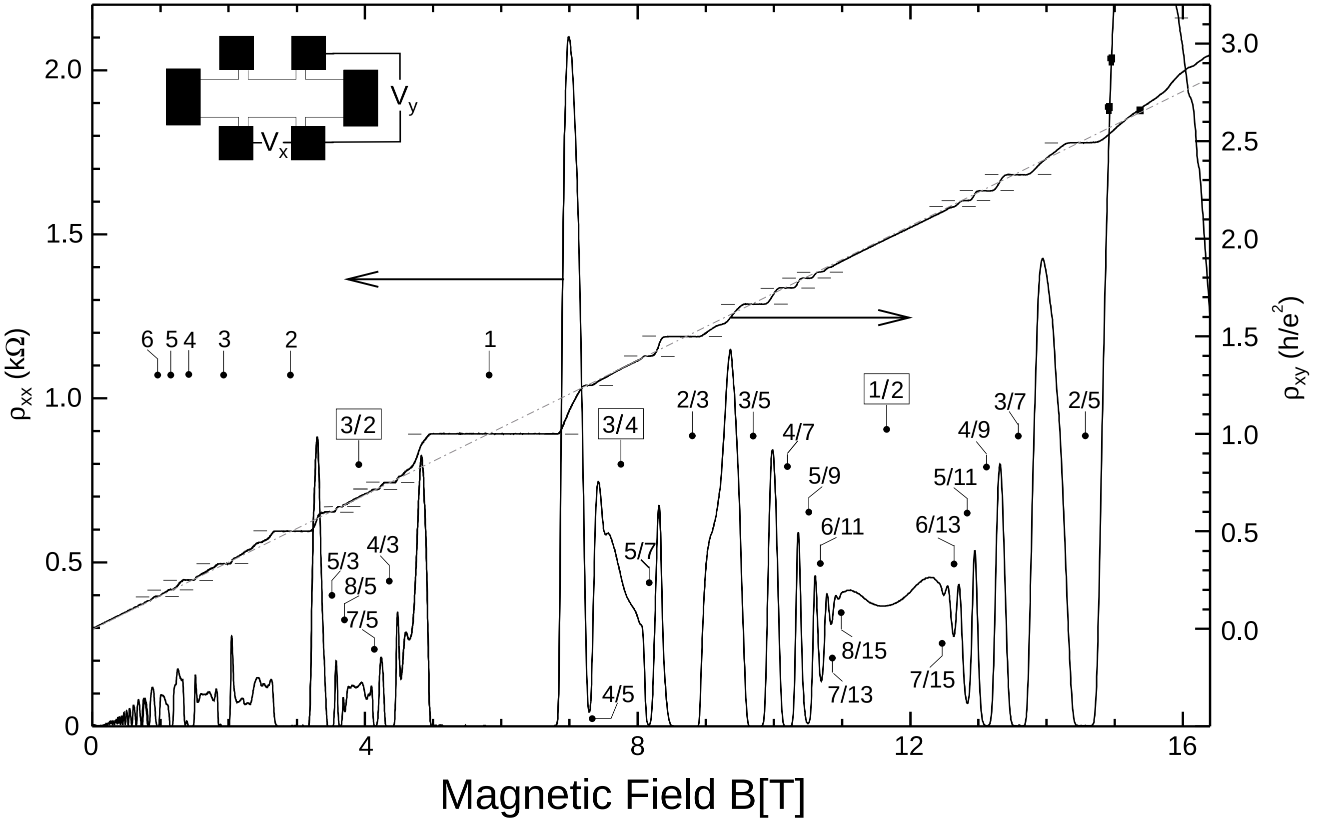
- symmetry breaking
  (a) broken (gauge) symmetry U(1)
  (b) broken (rotation) symmetry O(3)
- emergence of (collective) Goldstone modes
  (a) superfluid mode, with $\omega \propto |\mathbf{k}|$
  (b) spin waves, with $\omega \propto |\mathbf{k}|^2$

The Revolution(s) of the 1980ies

3 essential discoveries:

- integer quantum Hall effect (1980, v. Klitzing, Dorda, Pepper)
- fractional quantum Hall effect (1982, Tsui, Störmer, Gossard)
- high-temperature superconductivity (1986, Bednorz, Müller)

Integer Quantum Hall Effect (I)



[mesurement by J. Smet et al., MPI-Stuttgart]

QHE = plateau in Hall res. & vanishing long. res.

Integer Quantum Hall Effect (II)

Quantised Hall resistance at low temperatures

$$R_H = \frac{h}{e^2} \frac{1}{n}$$

h/e^2: universal constant
n: quantum number (topological invariant)

- result independent of geometric and microscopic details
- quantisation of high precision ($> 10^9$)

$\Rightarrow$ resistance standard: $R_{K-90} = 25\,812,807\,\Omega$

Fractional Quantum Hall Effect

partially filled Landau level $\rightarrow$ Coulomb interactions relevant

1983: Laughlin's N-particle wave function

- no (local) order parameter associated with symmetry breaking
- no Goldstone modes
- quasi-particles with fractional charges and statistics

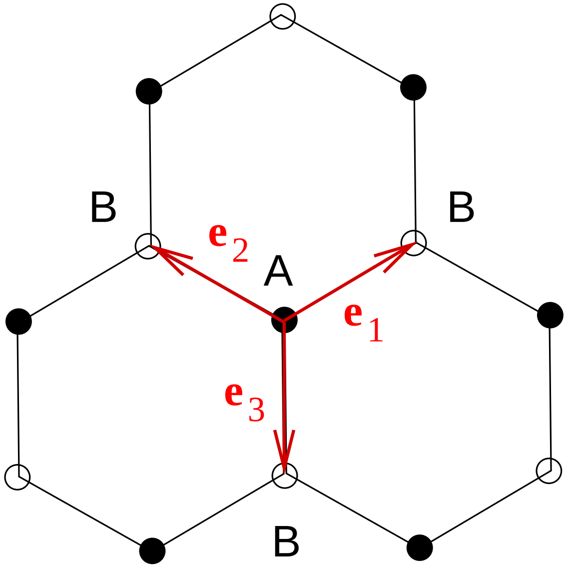
1990ies : description in terms of topological (Chern-Simons) field theories

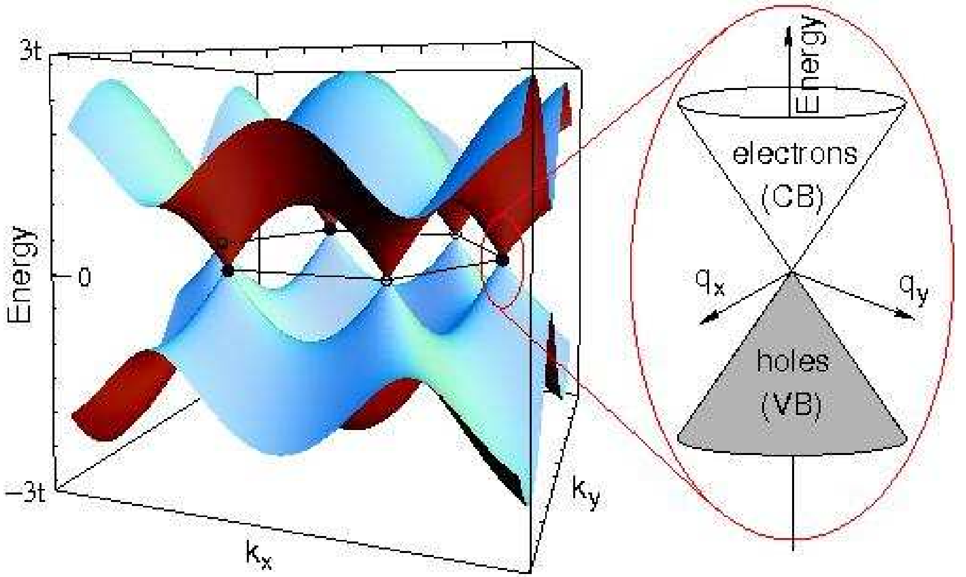
The Physics of the New Millenium

- simulation of condensed-matter models with optical lattices (cold atoms)
- 2004 : physics of graphene (2D graphite)
- 2005-07 : topological insulators

Graphene – First 2D Crystal

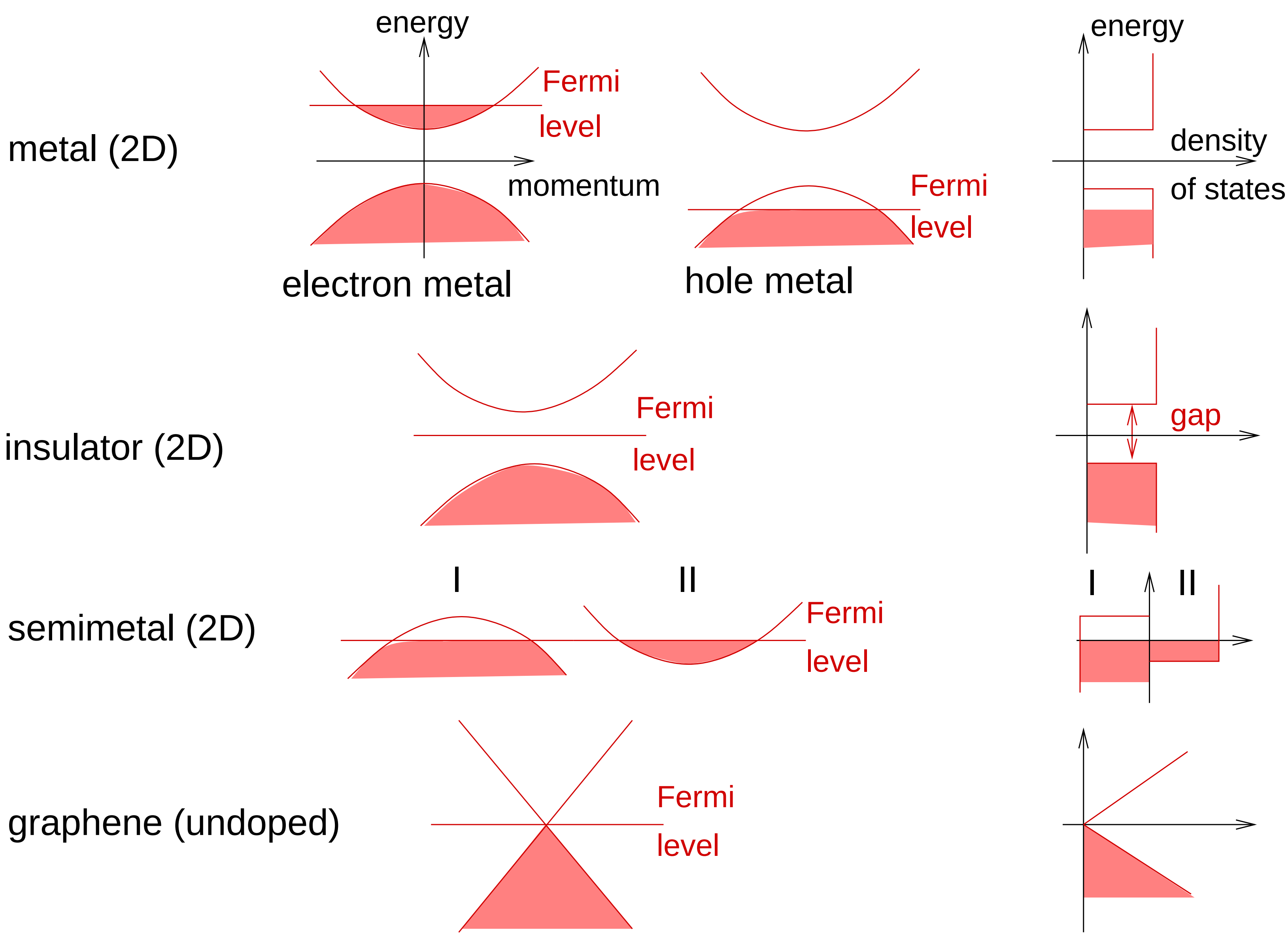
- honeycomb lattice =
  two triangular (Barvais) lattices





band structure

Band Structure and Conduction Properties (Bis)



Topological Insulators

generic form of a two-band Hamiltonian:

$$H = \epsilon_0(\mathbf{q})\mathbb{1} + \sum_{j=x,y,z} \epsilon_j(\mathbf{q})\sigma^j$$

- Haldane (1988): anomalous quantum Hall effect $\rightarrow$ quantum spin Hall effect (QSHE)
- Kane and Mele (2005): graphene with spin-orbit coupling
- Bernevig, Hughes, Zhang (2006): prediction of a QSHE in HgTe/CdTe quantum wells
- König *et al.* (2007): experimental verification of the QSHE

$\Rightarrow$ 3D topological insulators (mostly based on bismuth): surface states $\sim$ ultra-relativistic massless electrons

Outline of the Classes

Mon : Introduction and Landau quantisation

Tue : Issues of the IQHE; Introduction to the Berry phase

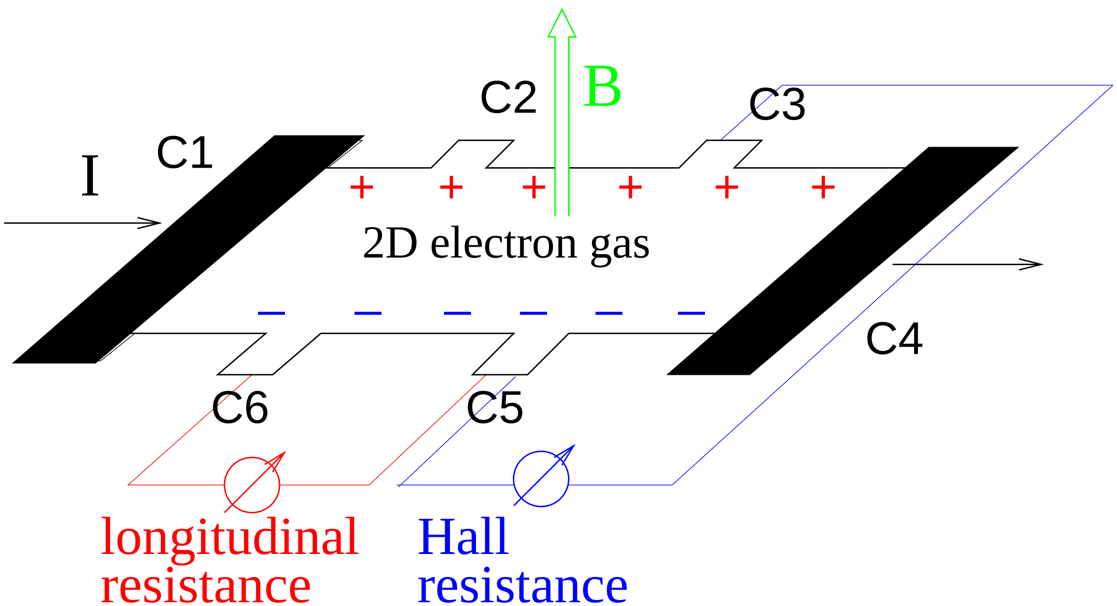
Thu : Simple models for topological insulators

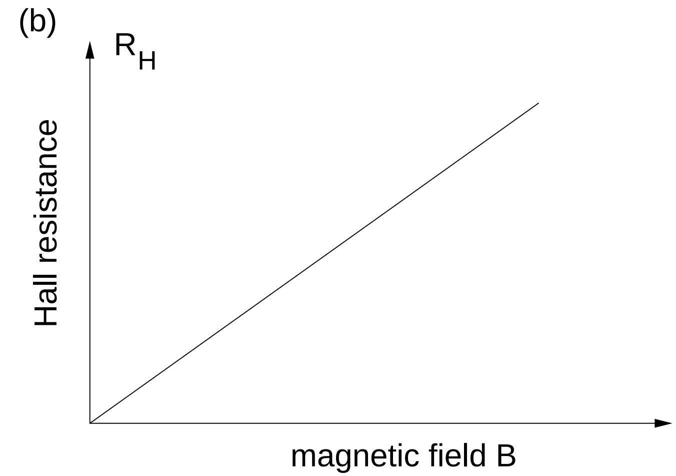
Further Reading

- D. Yoshioka, *The Quantum Hall Effect*, Springer, Berlin (2002).
- S. M. Girvin, *The Quantum Hall Effect: Novel Excitations and Broken Symmetries*, Les Houches Summer School 1998 http://arxiv.org/abs/cond-mat/9907002
- G. Murthy and R. Shankar, Rev. Mod. Phys. **75**, 1101 (2003). http://arxiv.org/abs/cond-mat/0205326
- M. O. Goerbig, *Quantum Hall Effects* http://arxiv.org/abs/0909.1998
- B. A. Bernevig, *Topological Insulators and Topological Superconductors*, Princeton UP (2013).

1. Introduction To the Integer Quantum Hall Effect and Materials

Classical Hall Effect (1879)





Hall resistance:

$$R_H = B/en_{el}$$

Quantum Hall system :
2D electrons in a B-field

Drude model (classical stationary equation):

$$\frac{d\mathbf{p}}{dt} = -e\left(\mathbf{E} + \frac{\mathbf{p}}{m} \times \mathbf{B}\right) - \frac{\mathbf{p}}{\tau} = 0$$

Resistivity and Conductivity

$$\begin{aligned} \sigma_0 E_x &= -\frac{en_{el}}{m} p_x - \frac{en_{el}}{m} p_y (\omega_C \tau) \\ \sigma_0 E_y &= \frac{en_{el}}{m} p_x (\omega_C \tau) - \frac{en_{el}}{m} p_y \end{aligned}$$

Ohm's law : $\mathbf{E} = \rho \mathbf{j}$ with current density $\mathbf{j} = -en_{el}\mathbf{v} = -en_{el}\mathbf{p}/m$

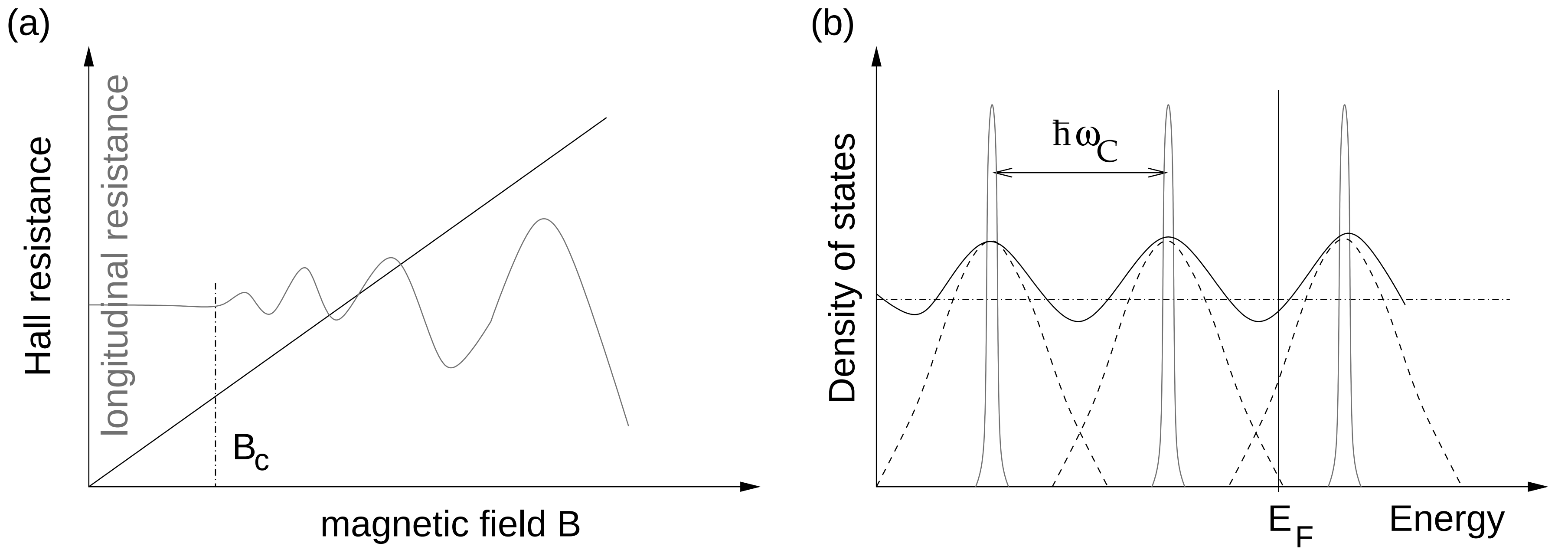
$\Rightarrow$ resistivity/conductivity tensor

$$\rho = \frac{1}{\sigma_0} \begin{pmatrix} 1 & \omega_C \tau \\ -\omega_C \tau & 1 \end{pmatrix} \qquad \sigma = \rho^{-1} = \frac{\sigma_0}{1 + (\omega_C \tau)^2} \begin{pmatrix} 1 & -\omega_C \tau \\ \omega_C \tau & 1 \end{pmatrix}$$

Link with mobility $\mu = e\tau/m$: $\omega_C \tau = \mu B$

Hall resistivity : $\rho_H = \omega_C \tau / \sigma_0 = B/en_{el}$

Shubnikov-de Haas Effect (1930)



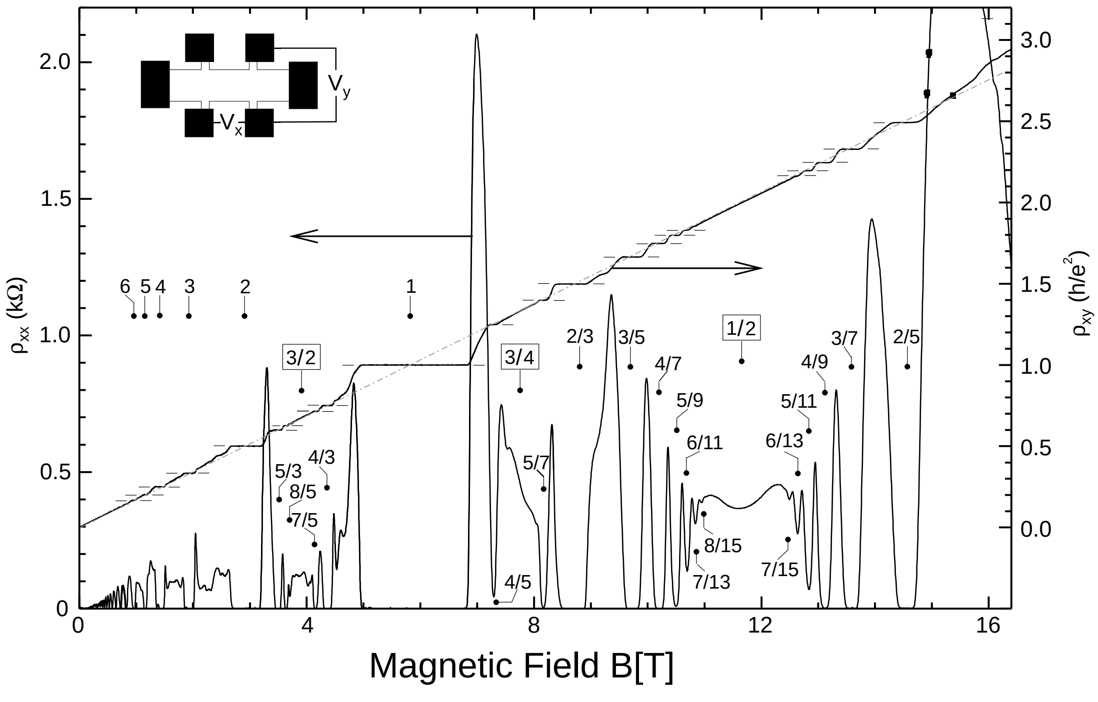
oscillations in longitudinal resistance

$\rightarrow$ Einstein relations $\sigma_0 \propto \partial n_{el}/\partial\mu \propto \rho(\epsilon_F)$

$\rightarrow$ Landau quantisation (into levels ϵ_n)

$$\sigma_0 \propto \rho(\epsilon_F) \propto \sum_n f(\epsilon_F - \epsilon_n)$$

Quantum Hall Effect (QHE)

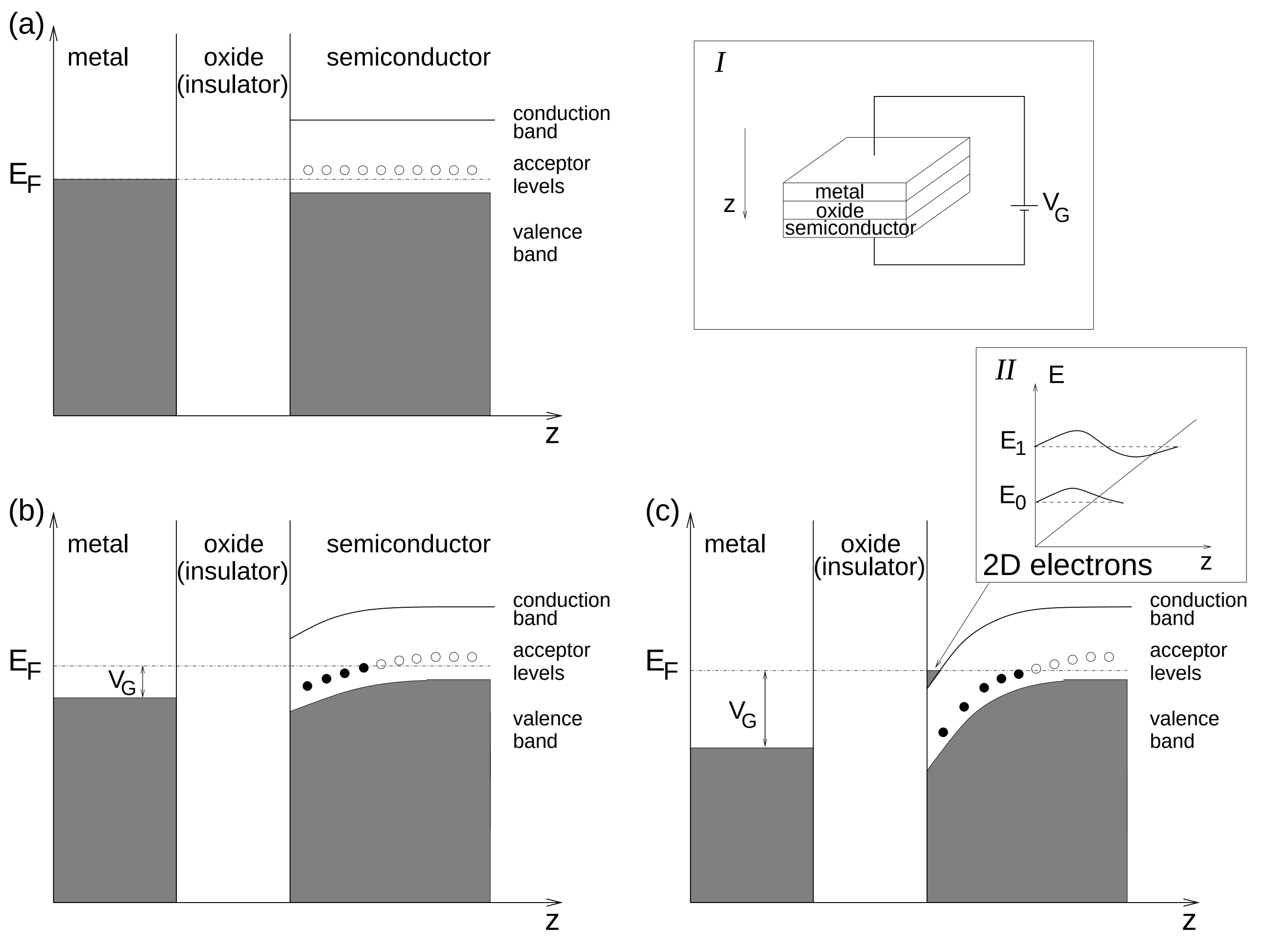


QHE = plateau in R_H & $R_L = 0$

1980 : Integer quantum Hall effect (IQHE)

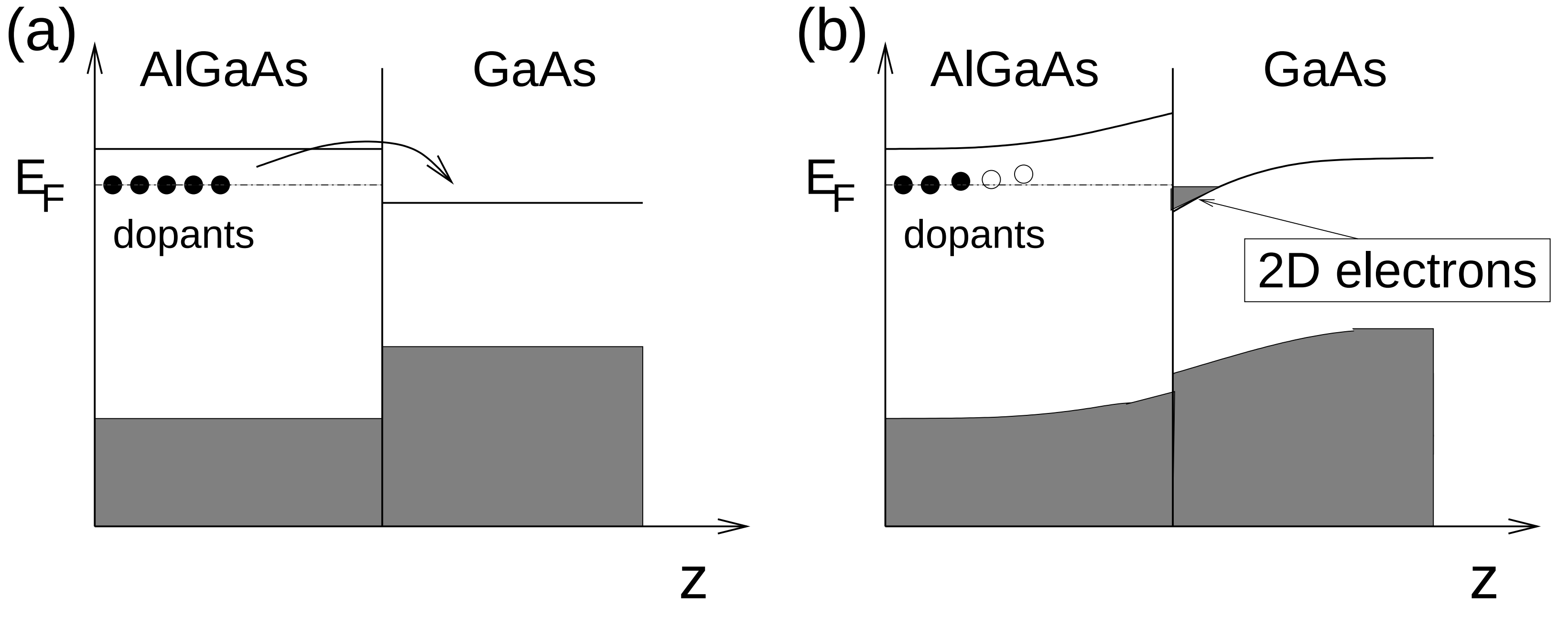
1982 : Fractional quantum Hall effect (FQHE)

Metal-Oxide Field-Effect Transistor (MOSFET)



usually silicon-based materials (Si/SiO_2 interfaces)

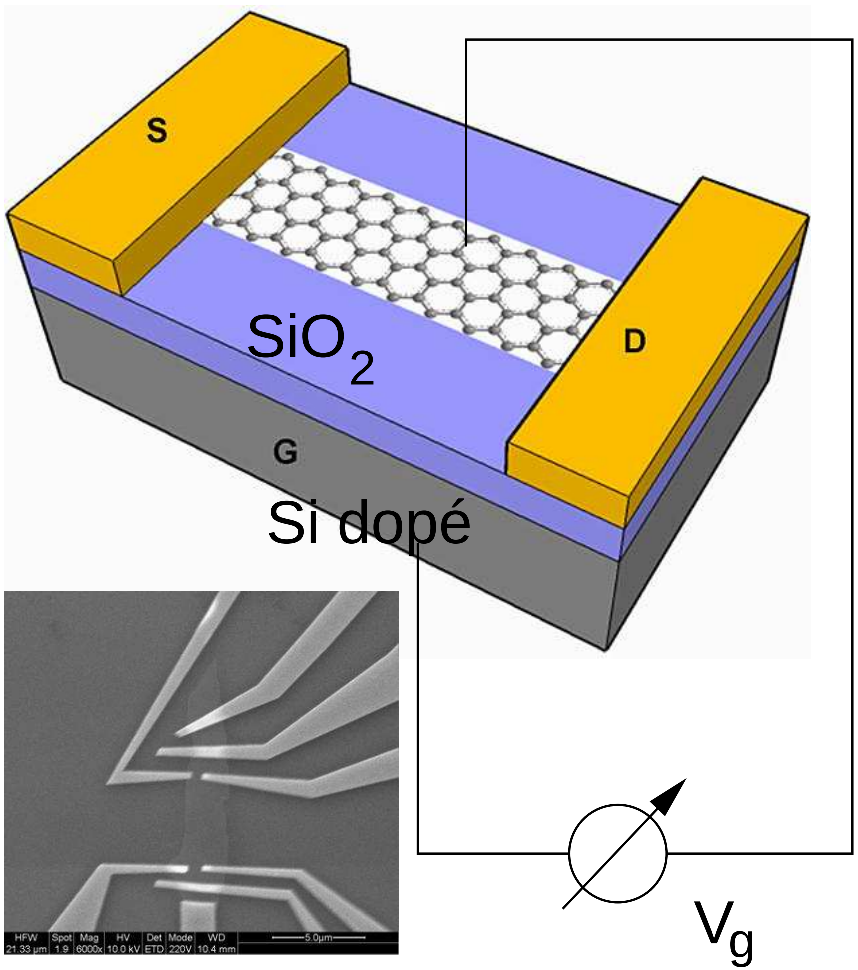
GaAs/AlGaAs Heterostructure

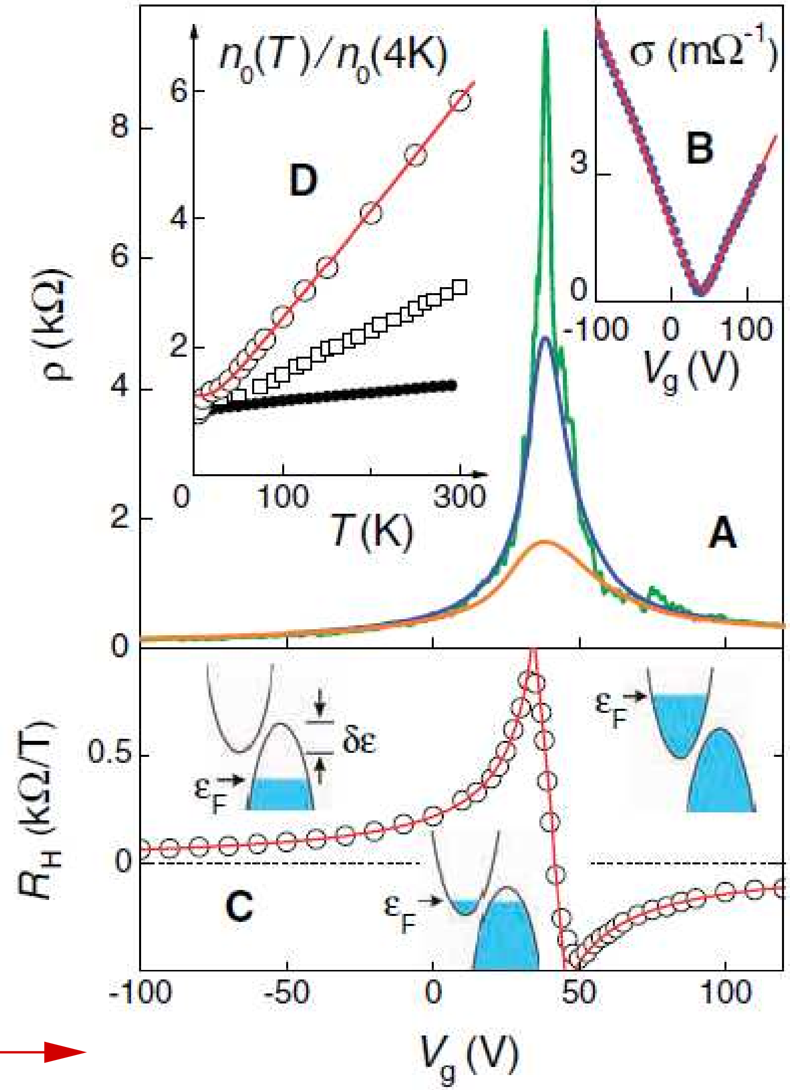


Impurity levels farther away from 2DEG (as compared to Si/SiO_2)

$\Rightarrow$ enhanced mobility (FQHE)

Electronic Mesurement of Graphene





Novoselov et al., Science 306, p. 666 (2004)

2. Landau Quantisation and Integer Quantum Hall Effect

Landau quantisation (reminder of first class)

- 2D electrons in continuum limit ($|\mathbf{q}|a \ll 1$)

$$H(p_x, p_y) \qquad \mathbf{p} = \hbar\mathbf{q}$$

- Peierls substitution : $\mathbf{p} \to \mathbf{\Pi} = \mathbf{p} + e\mathbf{A}(\mathbf{r})$ for $a/l_B \ll 1$

$$H(p_x, p_y) \qquad \to \qquad H_B(\Pi_x, \Pi_y)$$

- Quantum mechanics

$$[x, p_x] = [y, p_y] = i\hbar \qquad \to \qquad [\Pi_x, \Pi_y] = -i\frac{\hbar^2}{l_B^2}$$

$\Rightarrow$ Ladder operators $[a, a^\dagger] = 1$

$$\Pi_x = \frac{\hbar}{\sqrt{2}l_B}\left(a^\dagger + a\right) \qquad \Pi_y = \frac{\hbar}{\sqrt{2}il_B}\left(a^\dagger - a\right)$$

Simple Landau levels

Schrödinger fermions :

$$H_B|n\rangle = \epsilon_n|n\rangle, \qquad a^\dagger a|n\rangle = n|n\rangle, \qquad \epsilon_n = \hbar\omega_C(n+1/2)$$

Dirac fermions (graphene) :

$$H_B = \sqrt{2}\frac{\hbar v_F}{l_B}\begin{pmatrix} 0 & a \\ a^\dagger & 0 \end{pmatrix}, \qquad \psi_n = \begin{pmatrix} u_n \\ v_n \end{pmatrix}, \qquad \epsilon_{\lambda=\pm,n} = \lambda\hbar\frac{v_F}{l_B}\sqrt{2n}$$

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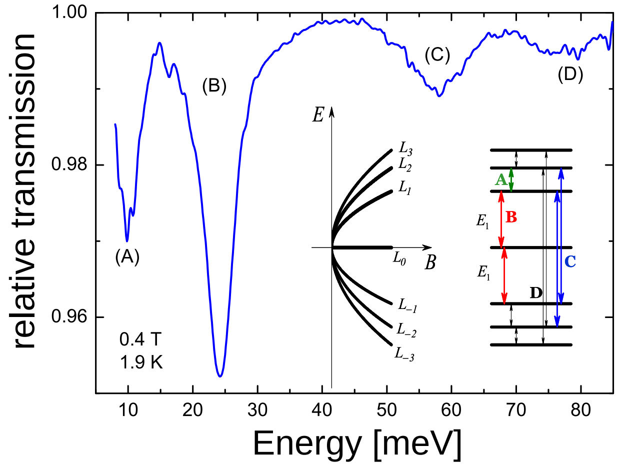
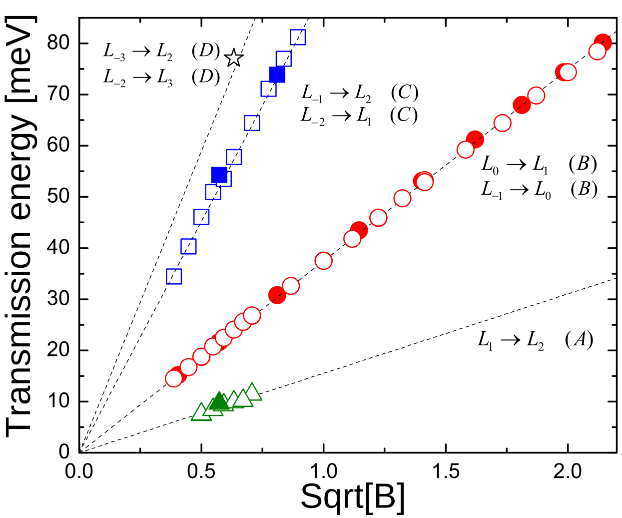
$$H_B = \sqrt{2}\frac{\hbar v_F}{l_B}\left(\begin{array}{cc} 0 & a \\ a^\dagger & 0 \end{array}\right), \qquad \psi_n = \left(\begin{array}{c} u_n \\ v_n \end{array}\right), \qquad \epsilon_{\lambda=\pm,n} = \lambda\hbar\frac{v_F}{l_B}\sqrt{2n}$$

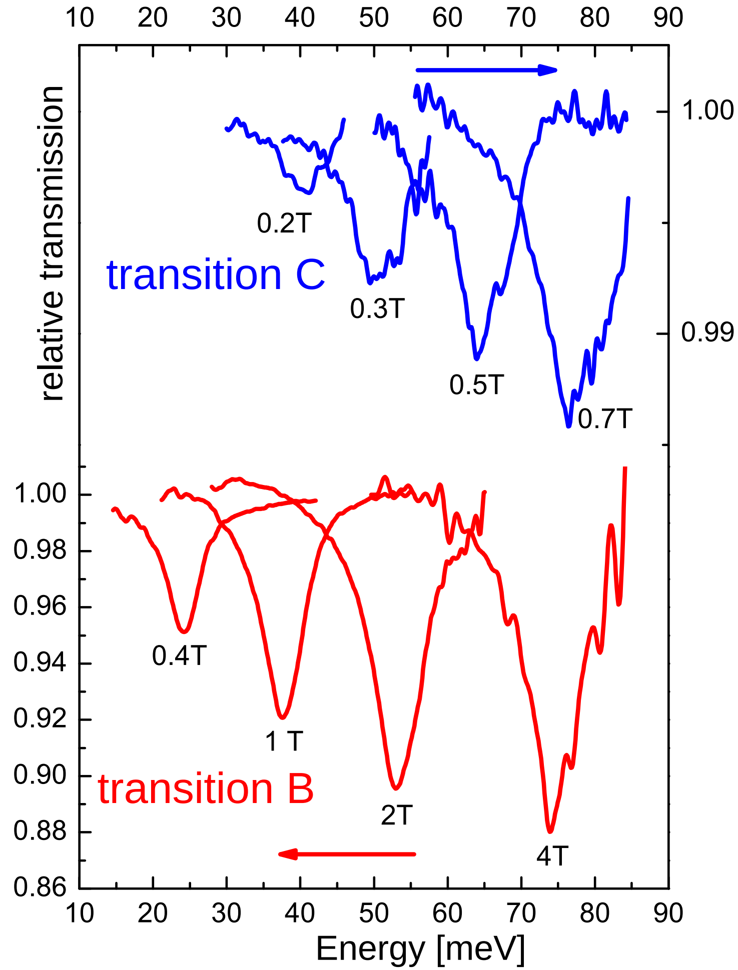
Eigenstates :

$$\psi_{n=0} = \left(\begin{array}{c} 0 \\ |n=0\rangle \end{array}\right) \qquad \psi_{\lambda,n=} = \frac{1}{\sqrt{2}}\left(\begin{array}{c} |n-1\rangle \\ \lambda|n\rangle \end{array}\right)$$

Infrared Transmission Spectroscopy





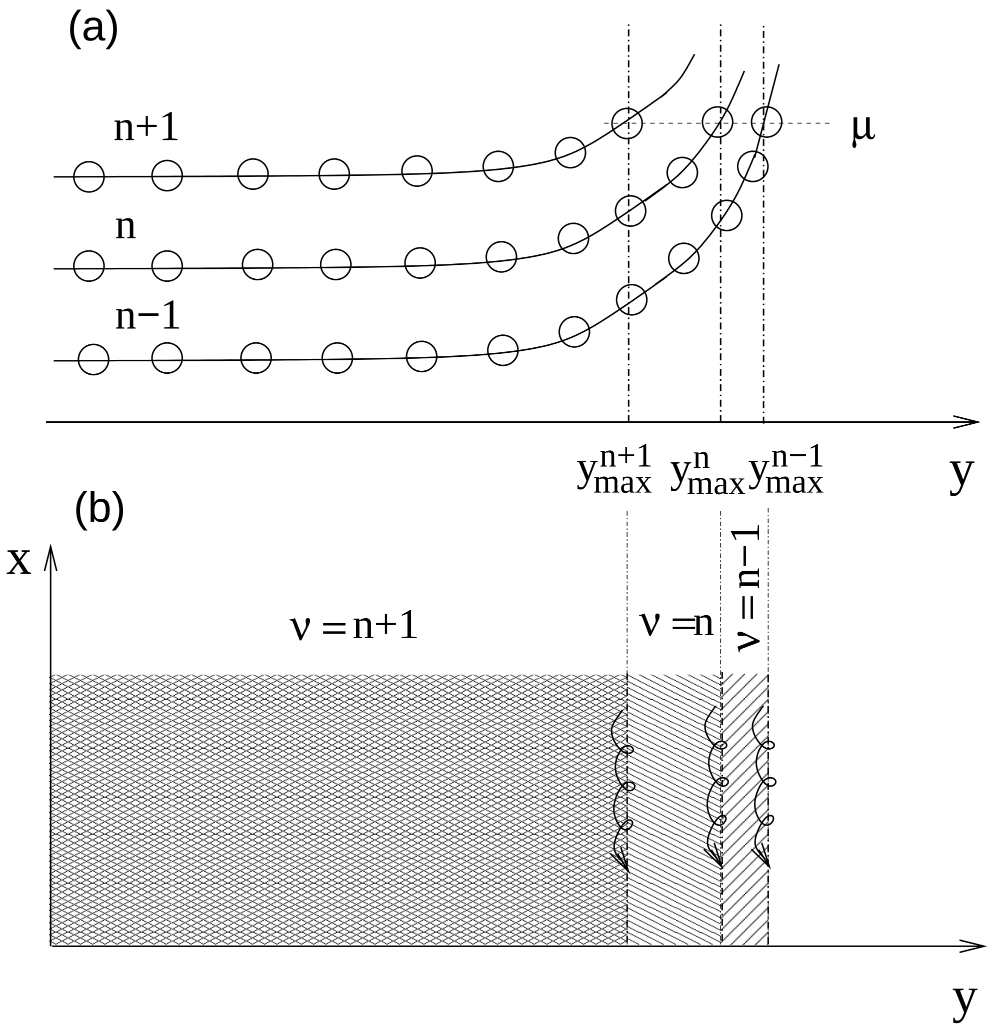


Grenoble high−field group: Sadowski et al., PRL 97, 266405 (2007)

selection rules :

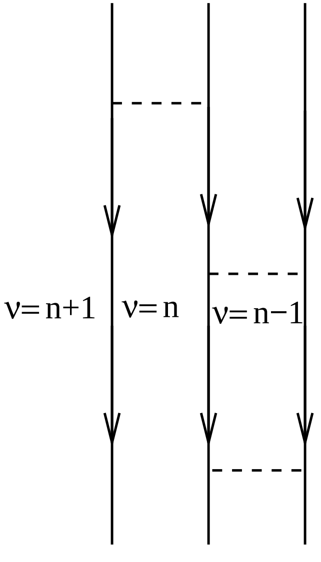
$$\lambda, n \rightarrow \lambda', n \pm 1$$

Edge States

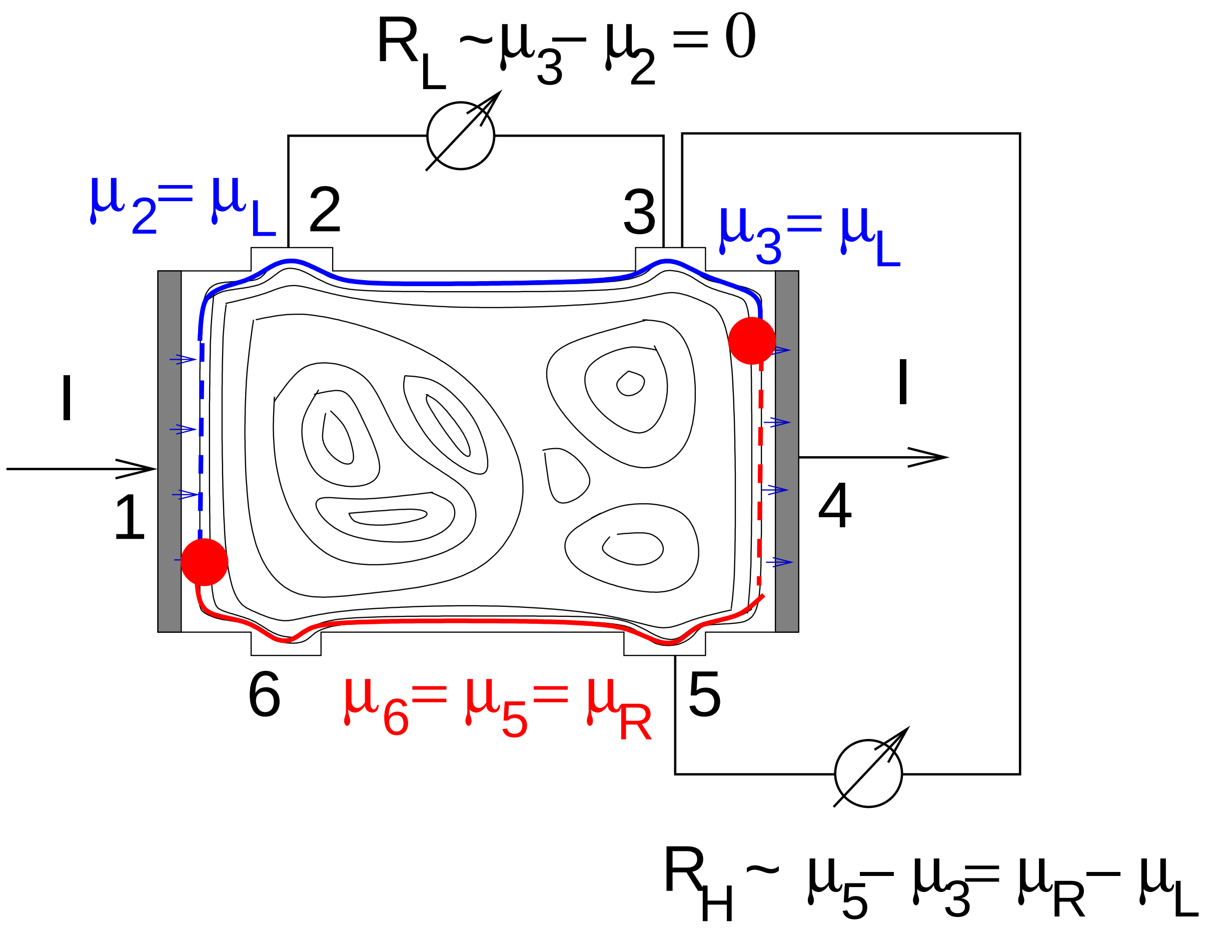


LLs bended upwards at the edges (confinement potential)

chiral edge states
⇒ only forward scattering



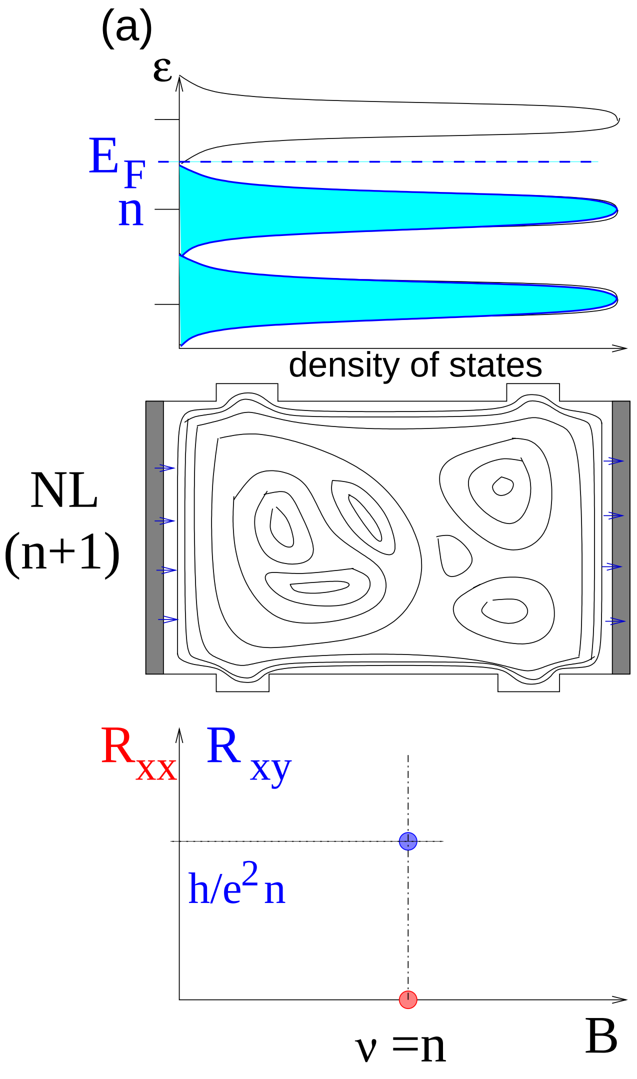
Four-terminal Resistance Measurement



● : hot spots

[Klass et al, Z. Phys. B:Cond. Matt. 82, 351 (1991)]

IQHE – One-Particle Localisation



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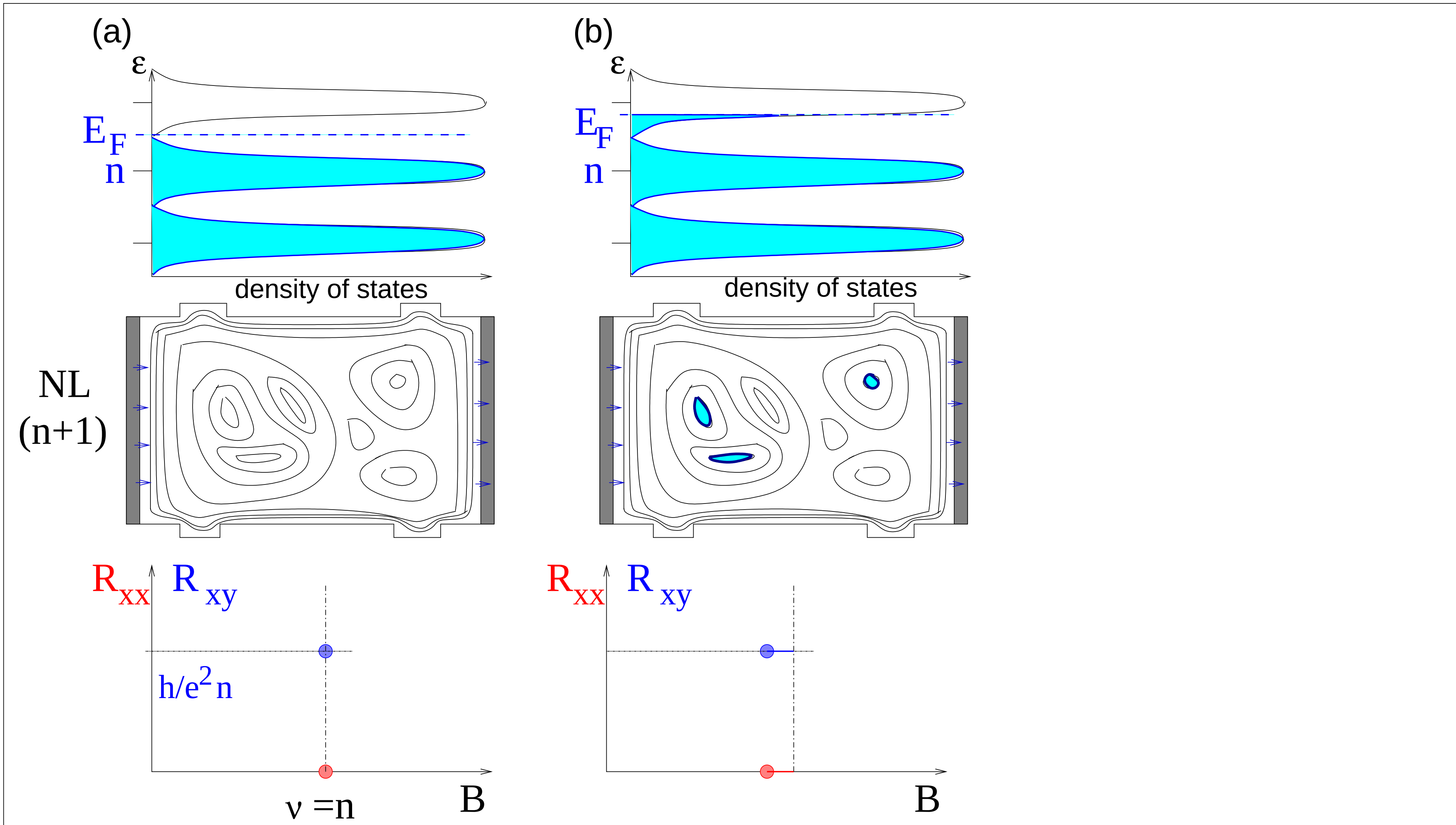
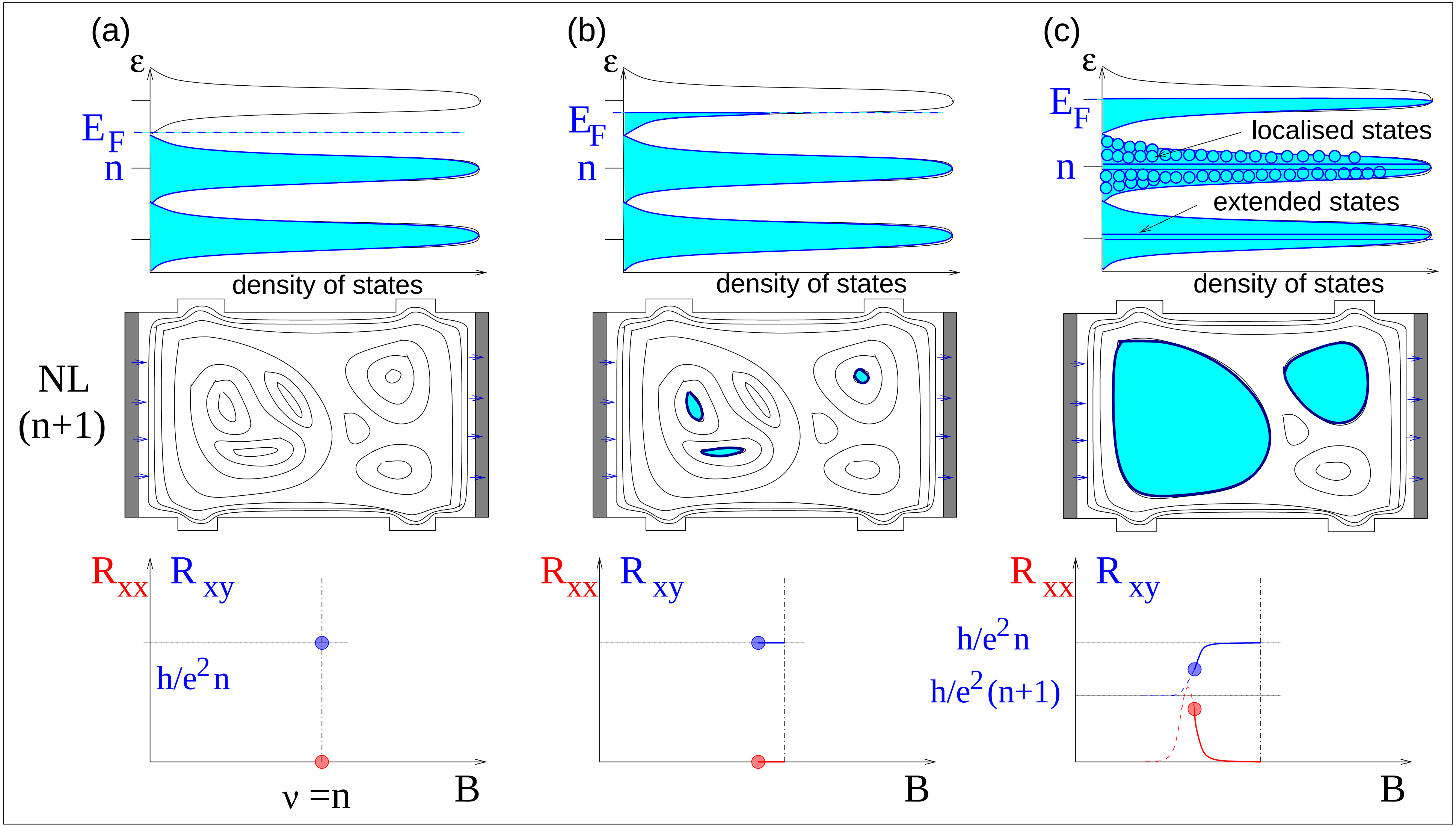


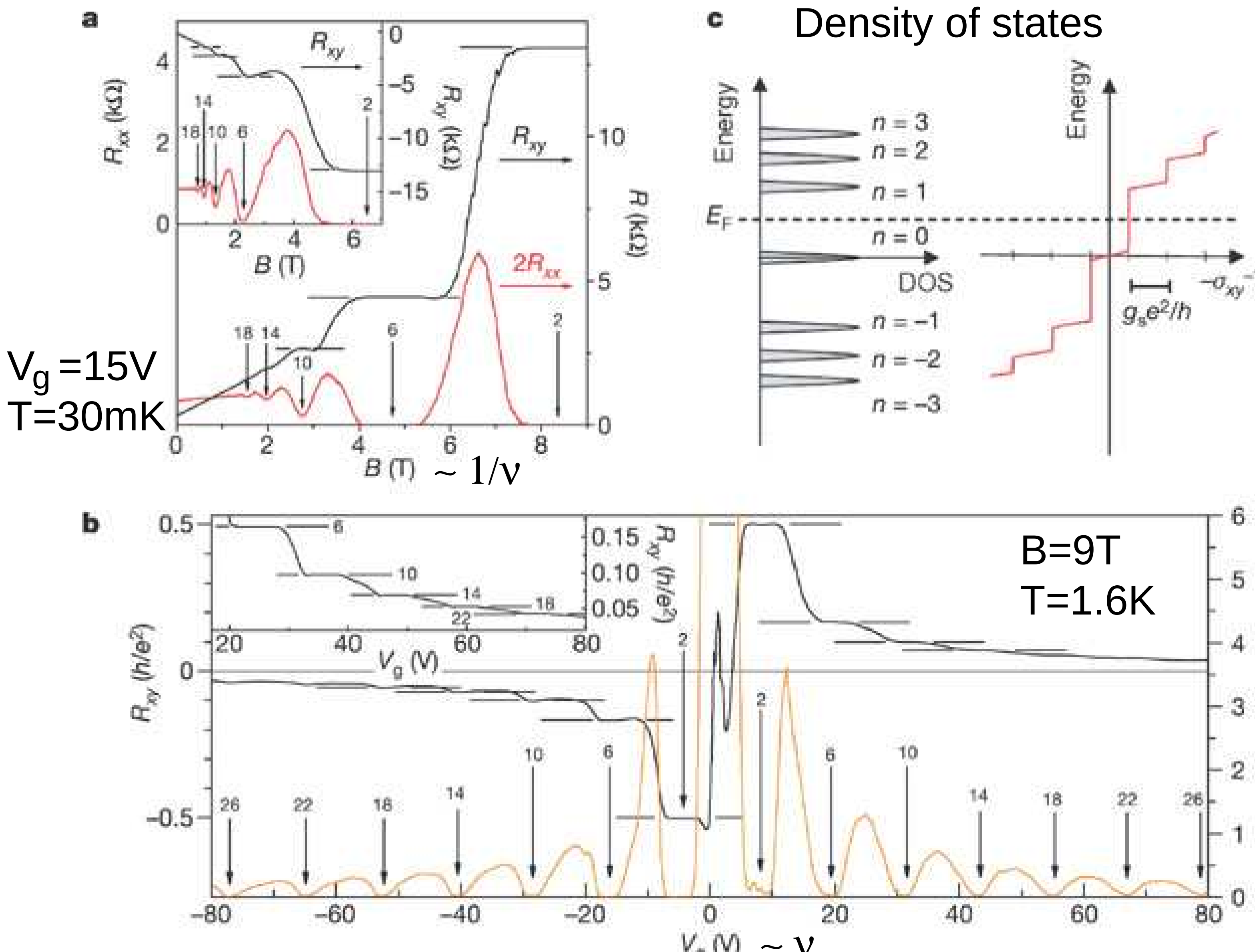
IQHE – One-Particle Localisation



IQHE in Graphene

Novoselov et al., Nature 438, 197 (2005)

Zhang et al., Nature 438, 201 (2005)



Graphene IQHE:

$R_H = h/e^2\nu$

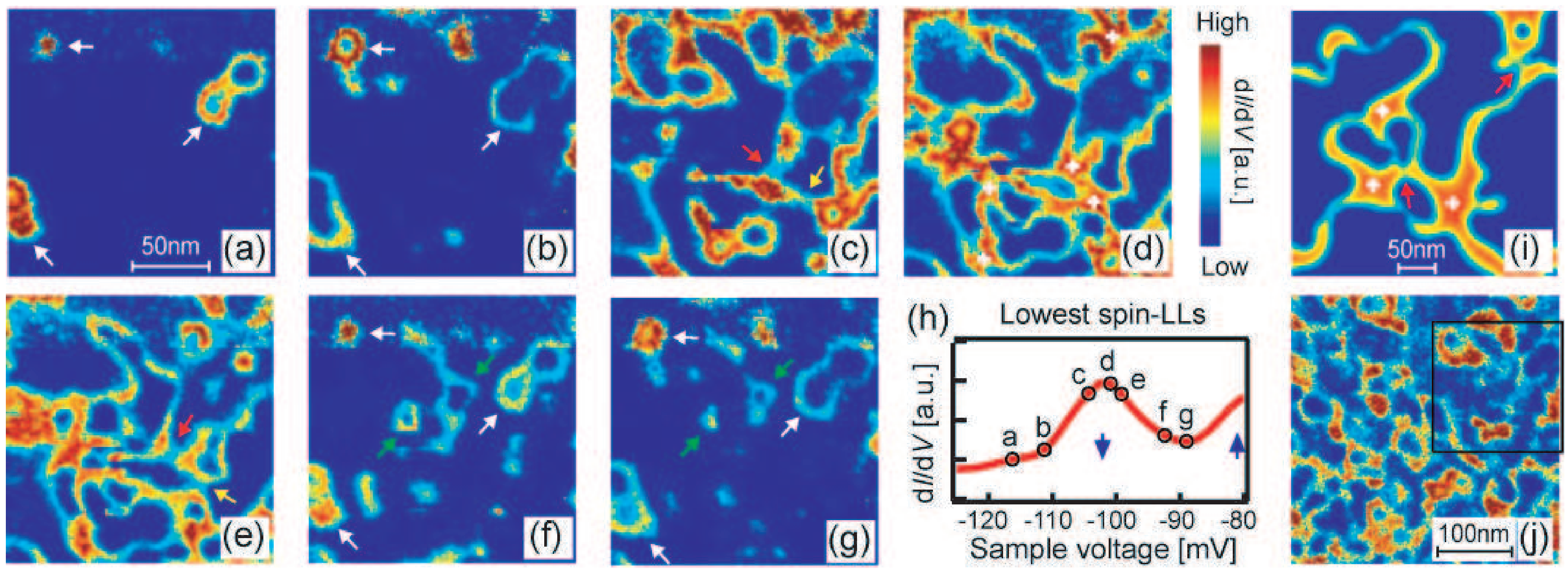
at $\nu = 2(2n+1)$

Usual IQHE:

at $\nu = 2n$

(no Zeeman)

Percolation Model – STS Measurement



2DEG on n-InSb surface Hashimoto et al., PRL 101, 256802 (2008)

(a)-(g) dI/dV for different values of sample potentials (lower spin branch of LL $n = 0$)

(i) calculated LDOS for a given disorder potential in LL $n = 0$

(j) dI/dV in upper spin branch of LL $n = 0$

Towards topological band theory

- Berry-ology :
  - Berry connexion : $\vec{\mathcal{A}}_n(\mathbf{k}) = i\psi_n^\dagger(\mathbf{k})\nabla_{\mathbf{k}}\psi_n(\mathbf{k})$
  - Berry curvature : $\vec{\mathcal{B}}_n(\mathbf{k}) = \nabla_{\mathbf{k}} \times \vec{\mathcal{A}}_n(\mathbf{k})$
  - Berry phase : $\gamma_n = \oint d\mathbf{k} \cdot \vec{\mathcal{A}}_n(\mathbf{k})$

$\rightarrow$ Chern number : $\mathcal{C}_n = \frac{1}{2\pi} \oint_{BZ} d\mathbf{k} \cdot \vec{\mathcal{A}}_n(\mathbf{k})$

Towards topological band theory

- Pseudo-Chern number of a (massive) Dirac point :

$$\tilde{\mathcal{C}} = \frac{1}{2}\xi \mathrm{sgn}(m)$$

Towards topological band theory

- Pseudo-Chern number of a (massive) Dirac point :

$$\tilde{\mathcal{C}} = \frac{1}{2}\xi \mathrm{sgn}(m)$$

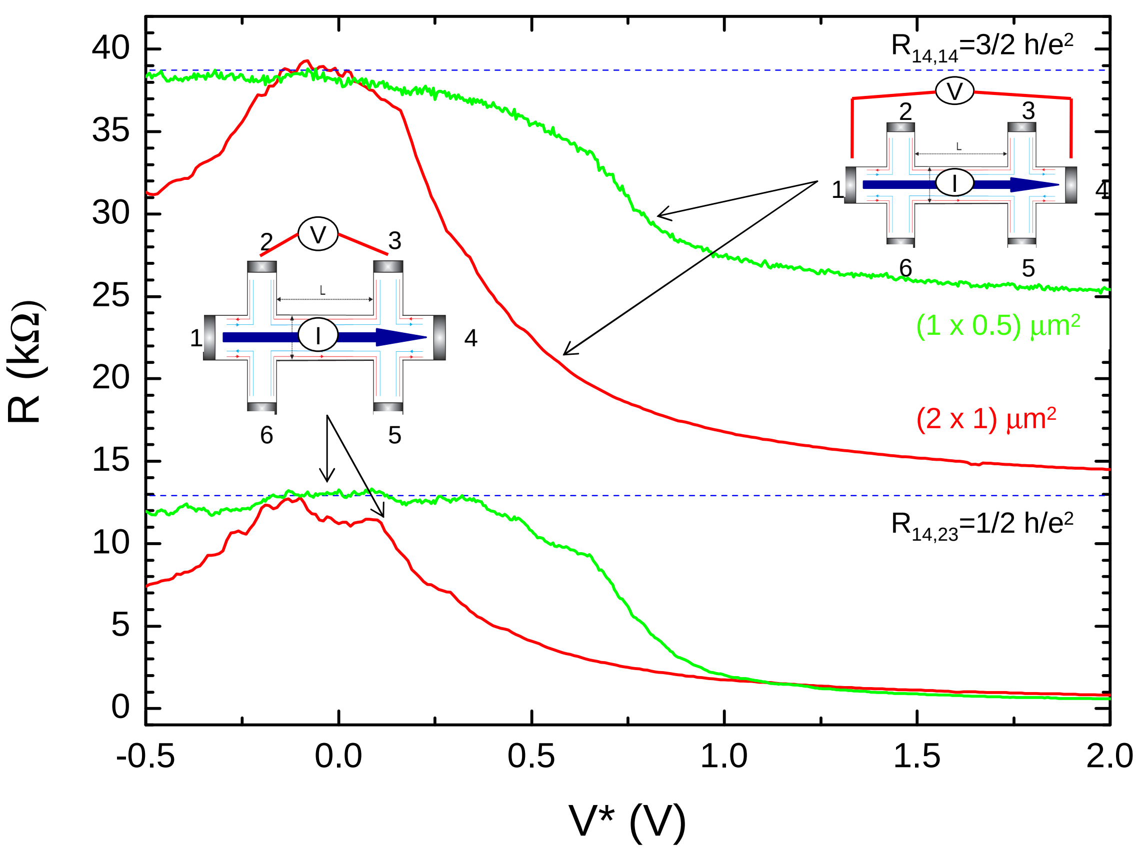
- Remarks :
  - half integer (not a true toplogical invariant) due to non-compact support $\mathbf{k} \in \mathbb{R}^2$
  - each Dirac point contributes $\pm 1/2$ to total Chern number

$\rightarrow$ Dirac points (on a lattice) come in pairs to get integer Chern numbers ! (fermion doubling)

  - Haldane model ($\tilde{\mathcal{C}}_K = \tilde{\mathcal{C}}_{K'}$) : $\mathcal{C} = \tilde{\mathcal{C}}_K + \tilde{\mathcal{C}}_{K'} = \pm 1$
  - Kane-Mele model : $\mathcal{C}_\uparrow = -\mathcal{C}_\downarrow$

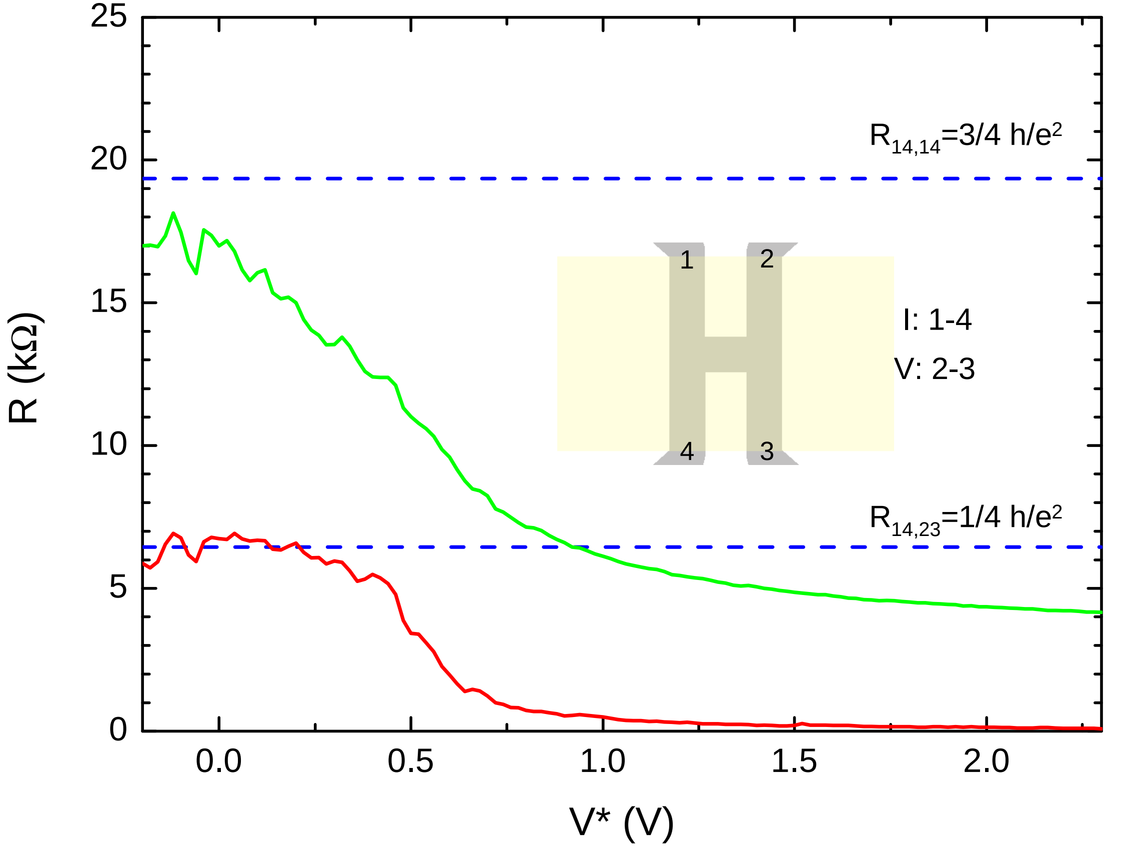
Non-local transport in QSHE (I)

CdTe/HgTe quantum wells [Roth et al., Science 2009]



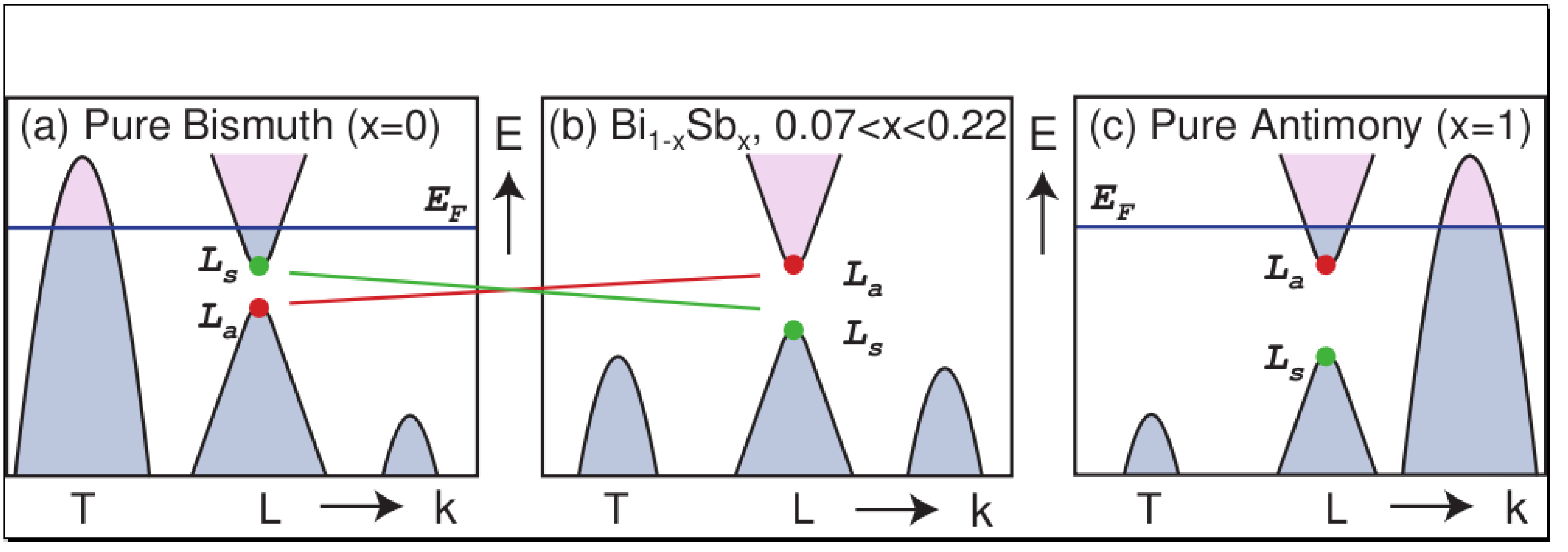
Non-local transport in QSHE l'EHQS (II)

CdTe/HgTe quantum wells [Roth et al., Science 2009]



3D topological insulators (I)

Frist generation based on Bi$_{1-x}$Sb$_x$ alloys [Hasan and Kane, RMP 2010]



→ band inversion above critical Sb concentration $x_c \simeq 0.04$

3D topological insulators (II)

- closing of ($\sim$ mass) gap at band inversion

$\Rightarrow$ Dirac fermions at surface of 3D toplogical insulator ($\sim$ edge states in 2D) :

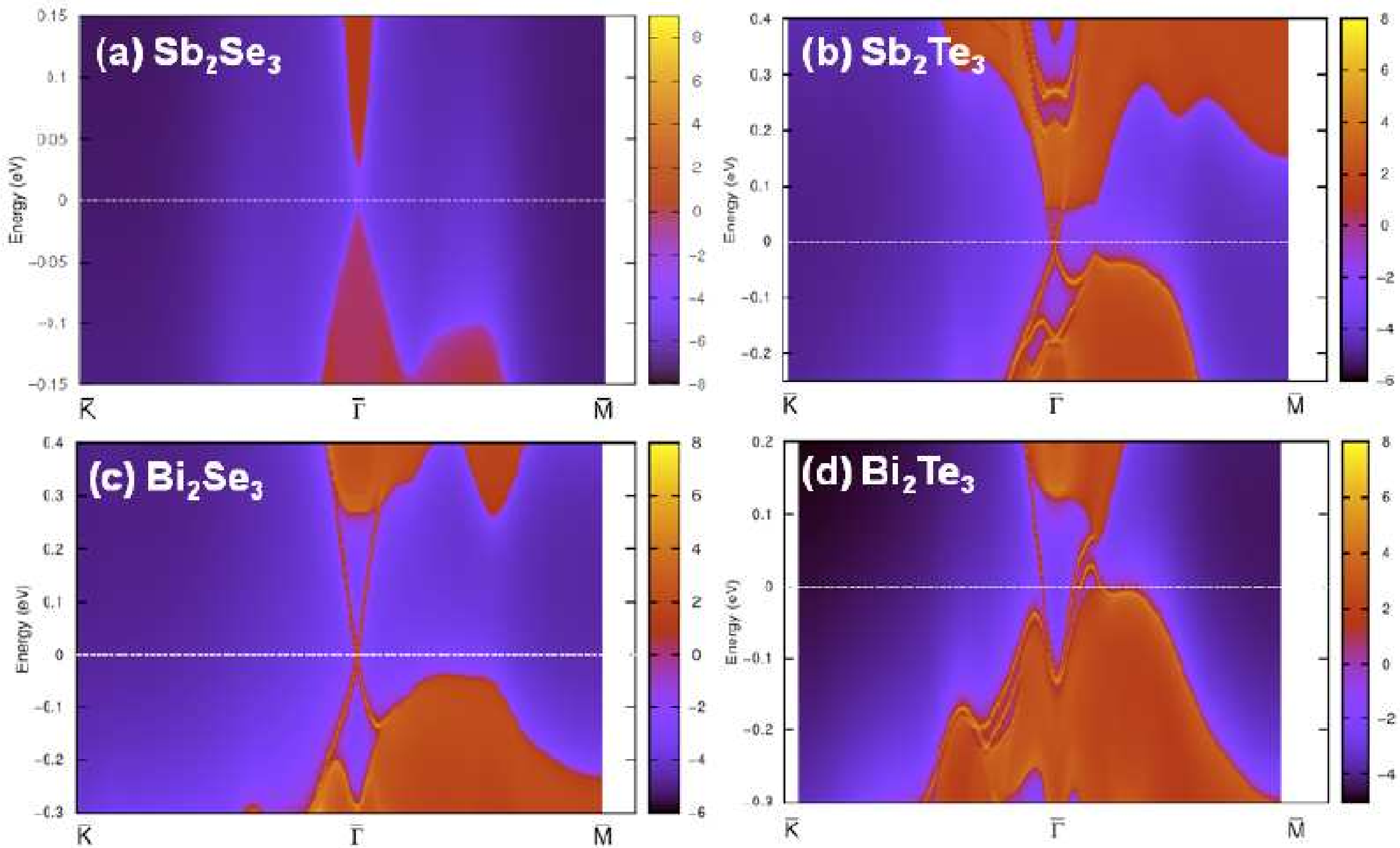
$$H_{\text{surface}} = v\mathbf{p} \cdot \boldsymbol{\sigma}$$

$\mathbf{p}$: momentum in surface
$\boldsymbol{\sigma}$: characterises *true* spin

$\Rightarrow$ single Dirac point (contrary to graphene with 4)

3D topological insulators (III)

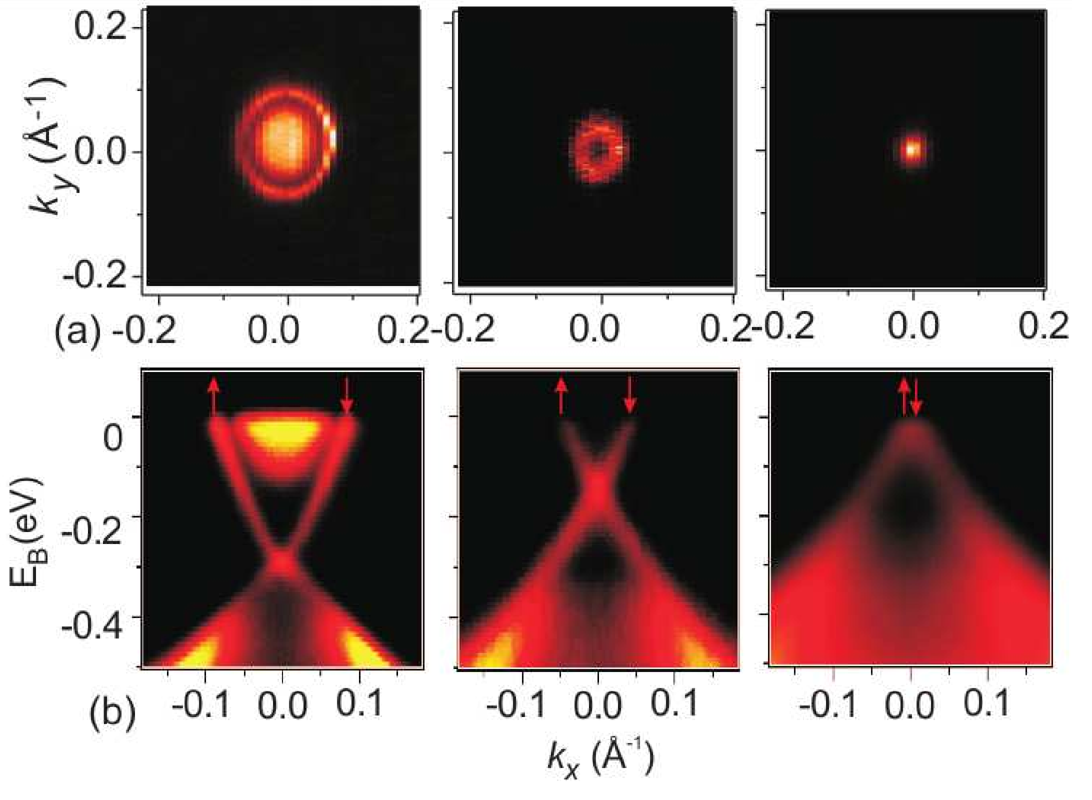
2^{nd} generation : Bi_2Se_3, Bi_2Te_2, Sb_2Te_3 [Zhang et al., 2009]



(*ab initio* calculations)

3D topological insulators (IV)

ARPES measurements of de Dirac fermions at a Bi_2Se_3 surface
[Hsieh et al., 2009]



$\rightarrow$ change of Fermi level by chemical doping (absorption of NO_2)