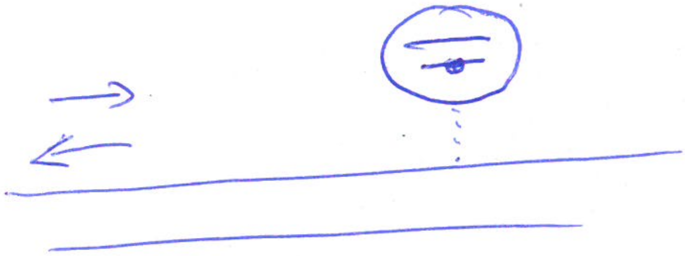
Scattering theory in RWA

• **RWA first:**



$$\underbrace{\Delta \sigma^+ \sigma^-}_{= \Delta |e\rangle\langle e|} = \Delta \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$H = \sum_{k\alpha} \omega_k \, a^\dagger_{k\alpha} a_{k\alpha} + \Delta \frac{\sigma^z}{2}$$
$$+ \frac{\sigma^+}{2} \sum_{k\alpha} g_k \, a_{k\alpha} + \frac{\sigma^-}{2} \sum_{k\alpha} g_k \, a^\dagger_{k\alpha}$$

$\alpha = L, R$ modes

with $\omega_k = v k \qquad k > 0$

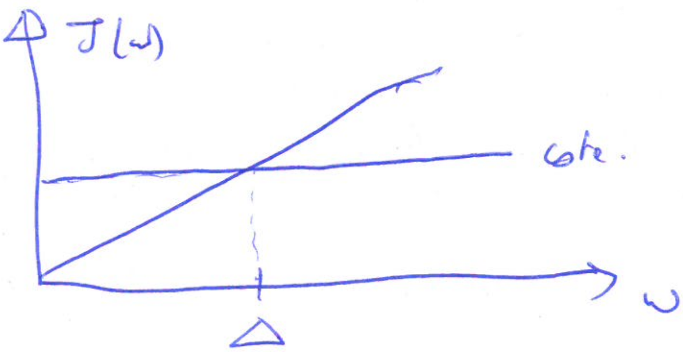
• **Local coupling:** Fermi Golden Rule

$$\Gamma = \frac{1}{4} J(\Delta) \qquad \text{for } \alpha \ll 1$$

$$\Gamma = \frac{\pi}{2} \alpha \Delta$$

Replace $\quad J(\omega) = 2\pi \alpha \omega e^{-\omega/\omega_p}$

by $\quad J(\omega) = \text{cst} = 4\Gamma = V^2$



$$\Rightarrow H = \sum_{k\alpha} \omega_k \, a^\dagger_{k\alpha} a_{k\alpha} + \Delta \, |e\rangle\langle e|$$
$$+ V \sigma^+ \int dx \, \delta(x) \sum_\alpha a^\dagger_\alpha(x) + h.c.$$

$$\boxed{\begin{aligned} H = & \sum_{k\alpha} \omega_k \, a^\dagger_{k\alpha} a_{k\alpha} + \Delta \, |e\rangle\langle e| \\ & + V \sigma^+ \sum_\alpha a^\dagger_\alpha(x=0) + h.c.. \end{aligned}}$$

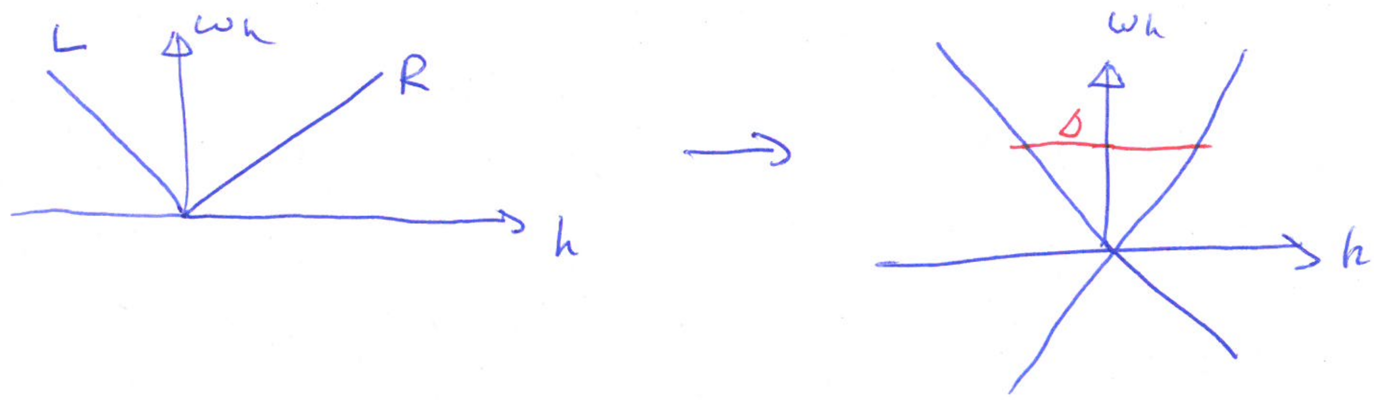
<u>Crucial point</u>: # excitations is conserved ?

N photons in input + qbit in $|g\rangle$

$\hookrightarrow$ { idem
OR
N-1 photons in input + qbit in $|e\rangle$

Of course energy can redistribute in complicated way (N-body scattering pb).

- <u>Linearize spectrum</u>:



Ok for physics near resonance if $\alpha \ll 1$

Ok only because RWA prevents boson to shower down to $-\infty$!!

$$H = \sum_{\substack{k\alpha \\ k\in\mathbb{R}}} k\, a^\dagger_{k\alpha} a_{k\alpha} + \Delta\, |e\rangle\langle e| + V\sigma^+ \sum_\alpha a^\dagger_\alpha(x=0) + h.c.$$

Here $\hbar = c = 1$

Fourier transform:

$$\boxed{\begin{aligned} H = &-i\int dx\, \left[a^\dagger_R(x)\,\partial_x a_R(x) + a^\dagger_L\,\partial_x a_L(x)\right] \\ &+ \Delta\, |e\rangle\langle e| + V\sigma^+\left(a_R(x=0) + a_L(x=0)\right) + h.c. \end{aligned}}$$

- Even-odd basis:

$$\begin{cases} a_e^\dagger(x) = \dfrac{a_R^\dagger(x) + a_L^\dagger(-x)}{\sqrt{2}} \\ a_o^\dagger(x) = \dfrac{a_R^\dagger(x) - a_L^\dagger(-x)}{\sqrt{2}} \end{cases}$$

$$\begin{cases} a_R^\dagger(x) = \dfrac{a_e^\dagger(x) + a_o^\dagger(-x)}{\sqrt{2}} \\ a_L^\dagger(x) = \dfrac{a_e^\dagger(x) - a_o^\dagger(-x)}{\sqrt{2}} \end{cases}$$

$\rightarrow H = H_e + H_o$

ST 4

$$\boxed{H_e = -i \int dx \quad a_e^\dagger \frac{\partial}{\partial x} a_e \quad + \quad \underbrace{V\sqrt{2}}_{V'}\sigma^+ a_e(x=0) + h.c. + \Delta |e\rangle\langle e|}$$

chiral

H_e = free

- **Scattering state:** in even basis

Input $|1 \text{ photon}\rangle \otimes |g\rangle$

energy k

$$|\psi_k\rangle = \int dx \quad \underbrace{e^{ikx}\left[\theta(-x) + t_k^e \theta(x)\right]}_{\psi_k(x)} \overbrace{a_e^\dagger(x)|0\rangle}|g\rangle$$

$$+ \varphi |0\rangle |e\rangle$$

~~No backscattering ...~~ $|t_k| \neq 1$ due to effect of emitter ! ← non chiral for $\psi_k(x)$

But $|t_k^e| = 1$ ← chiral !

- **Solve Schrödinger:**

$$H_e |\psi_k\rangle = k |\psi_k\rangle$$

$$-i \int dx \quad a_e^\dagger(x) \frac{\partial}{\partial x} a_e(x) \quad \int dy \quad \psi_k(y) a_e^\dagger(y) |0\rangle |g\rangle$$

$$= -i \int dx \int dy \quad \psi_k(y) |g\rangle a_e^\dagger(x) \frac{\partial}{\partial x} \underbrace{\left[a_e^\dagger(y) a_e(x) + \overbrace{\left[a_e(x), a_e^\dagger(y)\right]}^{\delta(x-y)} \right]}_{\text{zero !}} |0\rangle$$

$$\ldots = -i \int dx \int dy \; \psi_k(y) \; a_e^+(x) |0\rangle \frac{\partial}{\partial x} \delta(x-y) \; |g\rangle$$

$$= -i \int dx \;\; a_e^+(x) \; \frac{\partial \psi_k(x)}{\partial x} \;\; |0\rangle |g\rangle$$

Also: $\quad -i \int dx \; a_e^+ \frac{\partial}{\partial x} a \; \varphi |0\rangle |e\rangle = 0$

Also: $\quad \underbrace{\left[V' \sigma^+ a(x=0) + h.c. \right]}_{\to \text{ zero}} \int dx \; \psi_k(x) \; a_e^+(x) |0\rangle |g\rangle$

$$= V' |e\rangle \; \psi_k(x=0) \; |0\rangle$$

Also: $\quad \left[V' \sigma^+ \overbrace{a(x=0)}^{\text{zero}} + h.c. \right] \varphi |0\rangle |e\rangle$

$$= V' \varphi \; \overbrace{a^+(x=0)}^{\int dx \, a^+(x) \, \delta(x)} |0\rangle |g\rangle$$

Finally: $\quad \Delta \;\; |e\rangle\langle e| |\psi\rangle = \Delta \varphi |0\rangle |e\rangle$

$H|\psi\rangle = k |\psi\rangle$ gives two equations

$$\boxed{\begin{aligned} &\underline{|g\rangle \text{ part}}: \quad -i \frac{\partial \psi_k}{\partial x} + V' \varphi \, \delta(x) = k \, \psi_k(x) \\ &\underline{|e\rangle \text{ part}}: \quad \Delta \varphi + V' \psi_k(x=0) = k \varphi \end{aligned}}$$

Integrate the first one around zero $\quad$ ($\psi(0)$ not continuous)

$$\Rightarrow \quad -i \left[\psi_k(0^+) - \psi_k(0^-) \right] + V \varphi = 0$$

$$\Rightarrow \boxed{-i\left(t_k^e-1\right)+V'\varphi=0}$$

Plugin $\psi_k(x)$ into the second one:

$$\Rightarrow \Delta\varphi + V'\frac{1+t_k^e}{2} = k\varphi$$

$\theta(0)=\frac{1}{2}$

$$\Rightarrow \boxed{\varphi = V'\frac{1+t_k^e}{2}\frac{1}{k-\Delta}}$$

$$\Rightarrow -i\left(t_k^e-1\right)+V'^2\frac{1+t_k^e}{2}\frac{1}{k-\Delta}=0$$

$$2(k-\Delta)\left(t_k^e-1\right)+iV'^2\left(1+t_k^e\right)=0$$

$$t_k^e=\frac{2(k-\Delta)-iV'^2}{2(k-\Delta)+iV'^2}$$

$V'=\sqrt{2}V$

$$\boxed{t_k^e=\frac{k-\Delta-iV^2}{k-\Delta+iV^2}}$$

$\Rightarrow |t_k|=1$

chiral

in even channel

OK?

- Original basis:

$$a^+_R(x)=\frac{a^+_e(x)+a^+_o(-x)}{\sqrt{2}}$$

$a^+_o(-x)$ is a right mover?

$$\Rightarrow \psi_R(x)=\frac{1}{\sqrt{2}}\left(\psi_k(x)+e^{+ikx}\right)$$

$$\psi_k(x) = \frac{1}{\sqrt{2}}\left[e^{+ikx} + e^{ikx}\,\theta(-x) + e^{ikx}\,t_k^e\,\theta(x)\right]$$

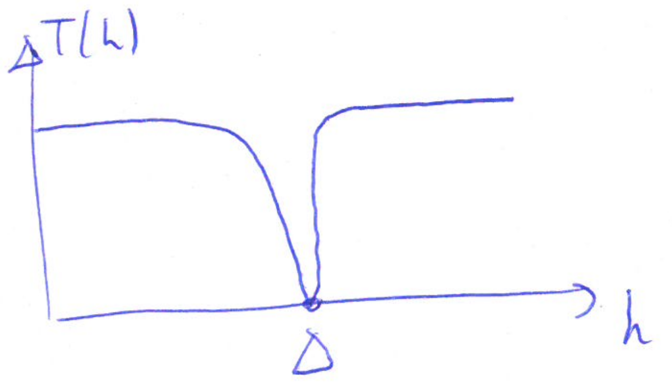
$$\psi_R(x) = \frac{1}{\sqrt{2}}\,e^{ikx}\left[(1+1)\,\theta(-x) + (1+t_k^e)\,\theta(x)\right]$$

$$= \sqrt{2}\,e^{ikx}\left[\theta(-x) + \left(\frac{1+t_k^e}{2}\right)\theta(x)\right]$$

$$\Rightarrow \boxed{t_k = \frac{1+t_k^e}{2}}$$

$$\Rightarrow \boxed{\boxed{t_k = \frac{k-\Delta}{k-\Delta+iV^2}}}$$

$$T = |t_k|^2 = \frac{(k-\Delta)^2}{(k-\Delta)^2+V^4} = \frac{(k-\Delta)^2}{(k-\Delta^0)^2+\Gamma^2}$$



Full absorpt° or resonance.

QBIT RENORMALIZATION

- Spin boson model in RWA :

$$H_{RWA} = \frac{\Delta}{2} \tau_z - \sum_k \frac{g_k}{2} \left(\tau^- a^+_k + \tau^+ a_k \right) + \sum_k \omega_k \, a^+_k a_k$$

Exact ground state :

$$|\psi\rangle = |\downarrow\rangle_{\tau_z} \otimes |0\rangle$$

$$|\psi\rangle = \frac{|\uparrow\rangle_{\sigma_z} - |\downarrow\rangle_{\sigma_z}}{\sqrt{2}} \otimes |0\rangle$$

- The full model at $\Delta = 0$:

$$H_{\Delta=0} = \sum_k \omega_k \, a^+_k a_k - \sigma_z \sum_k \frac{g_k}{2} \left(a^+_k + a_k \right)$$

$$H_{\Delta=0} = \sum_k \omega_k \left[a^+_k - \frac{\sigma_z g_k}{2\omega_k} \right] \left[a_k - \frac{\sigma_z g_k}{2\omega_k} \right] + \text{cst.}$$

Two degenerate vacua

$$|\psi_\uparrow\rangle = |\uparrow\rangle \, e^{\sum_k \frac{g_k}{2} (a^+_k - a_k)} |0\rangle$$

$$|\psi_\downarrow\rangle = |\downarrow\rangle \, e^{-\sum_k \frac{g_k}{2} (a^+_k - a_k)} |0\rangle$$

QR ①

Coherent states

$$|f\rangle = e^{fa^+ - f^* a}|0\rangle$$

$$|f\rangle \equiv D(f)|0\rangle$$

1) $\langle f|f\rangle = 1$ obviously

2) $D^+(f)\, a\, D(f) = a + f$

Proof: $D^+(f)\, a\, D(f) = e^{f^* a - f a^+}\, a\, e^{f a^+ - f^* a}$

$.. = a + [f^* a - f a^+, a]$ because commutator is constant

proof by Taylor expansion of Baker Campbell

$.. = a + f$ Can be proven in Fock basis

Idem $D^+(f)\, a^+ D(f) = a^+ + f^*$

3) $a|f\rangle = f|a\rangle$

$$a|f\rangle = a\, D(f)|0\rangle = D(f)\, \overbrace{D^+(f)\, a\, D(f)}^{a+f}|0\rangle$$

$$= D(f)\underbrace{(a+f)|0\rangle}_{\text{zero}} = f\, D(f)|0\rangle = f|f\rangle$$

4) $\langle f_1|f_2\rangle = \langle 0|D^+(f_1)\, D(f_2)|0\rangle$

$$= e^{-|f_1|^2 - |f_2|^2 + f_1^* f_2}$$

This derives again from Baker Campbell Hausdorf

$$e^{A+B} = e^A e^B e^{-\frac{1}{2}[A,B]} \quad \text{if } [A,B]$$

commutes with A and B..

Proof: $F(\lambda) = e^{\lambda(A+B)} \Rightarrow \frac{dF}{d\lambda} = (A+B)f$

$G(\lambda) = e^{\lambda A} e^{\lambda B} e^{\underbrace{-\frac{1}{2}[A,B]\lambda^2}_{\text{c-nr}}} \Rightarrow \frac{dG}{d\lambda} = AG + e^{\lambda A} B e^{\lambda B} e^{-\frac{1}{2}[A,B]\lambda^2} - [A,B]\lambda G$

$$\frac{dG}{d\lambda} = AG + \underbrace{e^{\lambda A} B e^{-\lambda A}}_{B+\lambda[A,B]} G - [A,B]\lambda G = (A+B)G$$

$G(0) = F(0) = 1 \Rightarrow G(\lambda) = F(\lambda) = 1$ ok!

· First trial wavefunct$^{\circ}$ for $\Delta \neq 0$

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left[|\uparrow\rangle \left| \frac{g}{2\omega} \right\rangle - |\downarrow\rangle \left| -\frac{g}{2\omega} \right\rangle \right]$$

$$|\psi\rangle = \frac{1}{\sqrt{2}} |\uparrow\rangle \prod_k e^{\frac{g_k}{2\omega_k} a^\dagger_k - h.c.} |0\rangle \quad - \quad (\,)$$

Tunnel energy:

$$\langle \sigma_x \rangle = \frac{1}{2} \langle \uparrow | \sigma_x | \downarrow \rangle \left\langle \frac{g}{2\omega} \middle| -\frac{g}{2\omega} \right\rangle + \text{item}$$

$$\langle \sigma_x \rangle = \left\langle \frac{g}{2\omega} \middle| -\frac{g}{2\omega} \right\rangle$$

$$\langle \sigma_x \rangle = e^{-2 \sum_k \left(\frac{g_k}{2\omega_k}\right)^2} = e^{-\frac{1}{2}\sum_k \frac{g_k^2}{\omega_k^2}}$$

$$\langle \sigma_x \rangle = e^{-\frac{1}{2} \int d\omega \sum_k \frac{g_k^2}{\omega^2} \delta(\omega - \omega_k)}$$

$$\langle \sigma_x \rangle = e^{-\frac{1}{2} \int \frac{d\omega}{\pi} \frac{J(\omega)}{\omega^2}}$$

$$\langle \sigma_x \rangle = e^{-\frac{1}{2} \int_0^{\omega_p} \frac{d\omega}{\pi} \frac{2\pi\alpha\omega}{\omega^2}}$$

$$\langle \sigma_x \rangle = e^{-\alpha \int_0^{\omega_p} d\omega \frac{1}{\omega}} = 0 \; ?.$$

orthogonality catastrophe ?

· Improve state:

$$|\Psi\rangle = \frac{|\uparrow\rangle|f\rangle - |\downarrow\rangle|-f\rangle}{\sqrt{2}}$$

with f_k variational parameter.

* $\langle\Psi|\Psi\rangle = 1$ satisfied

* $\langle\Psi|\frac{\Delta}{2}\sigma_x|\Psi\rangle = \frac{\Delta}{2} e^{-2\sum_k f_k^2}$ ← assume f_k real (exact)

* $\langle\Psi| -\sigma_z \sum_k \frac{g_k}{2}(a_k^\dagger + a_k)|\Psi\rangle$

$= -\frac{1}{2}\sum_k \frac{g_k}{2}\langle f|a_k^\dagger + a_k|f\rangle + \frac{1}{2}\sum_k \frac{g_k}{2}\langle -f|a_k^\dagger + a_k|-f\rangle$

$= -\frac{1}{2}\sum_k \frac{g_k}{2} 2f_k + \frac{1}{2}\sum_k \frac{g_k}{2}(-2f_k)$

$= -\sum_k g_k f_k$

* $\langle\Psi|\sum_k \omega_k a_k^\dagger a_k|\Psi\rangle = \frac{1}{2}\sum_k \omega_k \langle f|a_k^\dagger a_k|f_k\rangle$
$+ \frac{1}{2}\sum_k \omega_k \langle -f|a_k^\dagger a_k|-f\rangle$

$= \frac{1}{2}\sum_k \omega_k f_k^2 + \frac{1}{2}\sum_k \omega_k f_k^2 = \sum_k \omega_k f_k^2$

$$\langle\Psi|H|\Psi\rangle = E = \left.\sum_k \omega_k f_k^2 - \sum_k g_k f_k\right\} \text{classical elastic energy}$$

$$\left. + \frac{\Delta}{2} e^{-2\sum_k f_k^2}\right\} \text{tunnel energy}$$

Minimize :

$$\frac{dE}{df_k} = 2\omega_k f_k - g_k - 2\Delta f_k e^{-2\sum_q f_q^2}$$

$$\boxed{\begin{aligned} f_k &= \frac{g_k/2}{\omega_k + \Delta_R} \\ \Delta_R &= \Delta e^{-2\sum_q f_q^2} \end{aligned}}$$

cuts the divergence.

$$\Delta_R = \Delta \, e^{-2\sum_q \frac{(g_q/2)^2}{(\omega_q + \Delta_R)^2}}$$

$$\Delta_R = \Delta \, e^{-\frac{1}{2}\int d\omega \sum_q g_q^2 \delta(\omega - \omega_q) \frac{1}{(\omega + \Delta_R)^2}}$$

$$\Delta_R = \Delta \, e^{-\frac{1}{2}\int \frac{d\omega}{\pi} \frac{J(\omega)}{(\omega + \Delta_R)^2}} = \Delta \, e^{-\frac{1}{2}\int_0^{\omega_p} \frac{d\omega}{\pi} \frac{2\pi\alpha\omega}{(\omega + \Delta_R)^2}}$$

$$\Delta_R = \Delta \, e^{-\alpha \int_0^{\omega_p} d\omega \frac{\omega}{(\omega + \Delta_R)^2}} = \Delta e^{-\alpha \int_0^{\omega_p} d\omega \frac{\omega + \Delta_R - \Delta_R}{(\omega + \Delta_R)^2}}$$

$$\Delta_R = \Delta \, e^{+\alpha \log \frac{\Delta_R}{\Delta_R + \omega_p} + \pi\alpha\Delta_R \left(\frac{1}{\omega_p + \Delta_R} + \frac{1}{\Delta_R} \right)}$$

$$\omega_p \gg \Delta_R \Rightarrow \Delta_R = \Delta e^{+\alpha \log \frac{\Delta_R}{\omega_p} + \alpha}$$

$$\Delta_R = \Delta \left(\frac{\Delta_R}{\omega_p} \right)^{\alpha} e^{+\alpha}$$

$$\Delta_R^{1-\alpha} = \frac{\Delta e^{\alpha}}{\omega_p^{\alpha}}$$

$$\Delta_R = \left(\frac{\Delta e^{\alpha}}{\omega_p^{\alpha}}\right)^{\frac{1}{1-\alpha}} = \Delta\left(\frac{\Delta e}{\omega_p}\right)^{\frac{\alpha}{1-\alpha}}$$

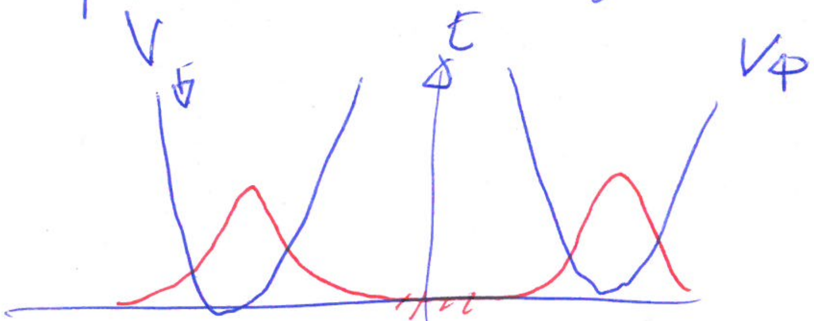
$$\boxed{\Delta_R = \Delta\left(\frac{\Delta e}{\omega_p}\right)^{\frac{\alpha}{1-\alpha}}}$$

For small α :

$$\Delta_R = \Delta\left[1 - \alpha \log \frac{\Delta e}{\omega_p}\right]$$

- Physical picture :

qbit dressed by coherent state with opposite displacement



small overlap $\Rightarrow$ tunneling is hampered.

- Entanglement :

$$|\psi\rangle = \frac{|\uparrow\rangle|f\rangle - |\downarrow\rangle|-f\rangle}{\sqrt{2}}$$

$$\hat{\rho} = |\psi\rangle\langle\psi|$$

$$\hat{\rho}_h = \underset{\text{All Modes} \neq h}{\mathrm{Tr}_{\text{Spin}}}\ \rho = \frac{1}{2}\left[|f_h\rangle\langle f_h| + |-f_h\rangle\langle -f_h|\right]$$

$$\hat{\rho}_k^2 = \frac{1}{4}\left[|\beta_k\rangle\langle\beta_k| + |-\beta_k\rangle\langle-\beta_k| + |\beta_k\rangle\langle-\beta_k| + |-\beta_k\rangle\langle\beta_k|\right]$$

$$\mathrm{Tr}\,\hat{\rho}_k^2 = \frac{1}{4}\left[1 + 1 + \langle\beta_k|-\beta_k\rangle + \langle-\beta_k|\beta_k\rangle\right]$$

$$\mathrm{Tr}\,\hat{\rho}_k^2 = \frac{1+e^{-2\beta_k^2}}{2}$$

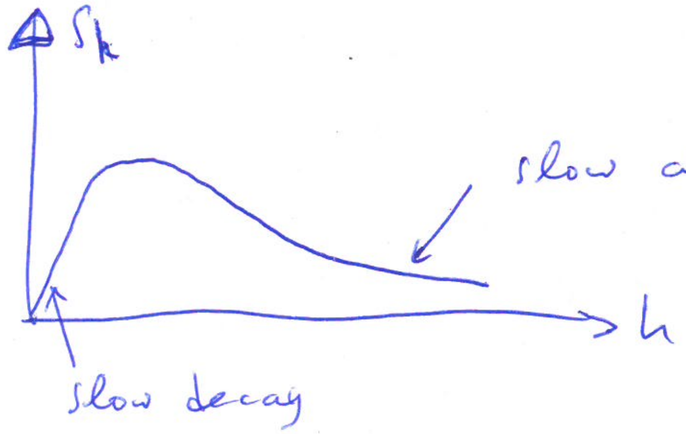
Renyi entropy (zero for pure state)

$$S_k = 1 - \mathrm{Tr}\,\hat{\rho}_k^2$$

$$S_k = \frac{1-e^{-2\beta_k^2}}{2}$$

$$\beta_k = \frac{g_k/2}{\omega_k+\Delta_k} = \frac{1}{2}\frac{\sqrt{2\alpha k}}{k+\Delta_R} \quad \text{for linear spectrum}$$

$$S_k = \frac{1}{2}\left[1 - e^{-\frac{\alpha k}{(k+\Delta_k)^2}}\right]$$



That's why it's tough !!

. Connection to Kondo :

$$\Delta_R = \Delta\left(\frac{e\Delta}{\omega_c}\right)^{\frac{\alpha}{1-\alpha}}$$

For $\alpha = 1 - j \qquad (j \ll 1)$

$$\Delta_R \simeq \Delta\left(\frac{e\Delta}{\omega_c}\right)^{1/j} \quad //$$

NON PERTURBATIVE POINT.

MQT $\Leftrightarrow$ Giant Lamb Shift $\Leftrightarrow$ Kondo