Non-linearities in quantum circuits

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“A gentle journey into not-so-gentle quantum many-body problems”
Plan and purpose of the lecture
Plan and purpose of the course

Organization of the lectures

Superconducting nanocircuits

► Lecture 1: i) General circuitry; ii) From Josephson junctions chains to spin-boson model

► Lecture 2: i) Scattering theory with dynamical emitter; ii) Frequency conversion beyond RWA; iii) Renormalization from many-body entanglement; iv) Many-body inelastic scattering theory

Metallic nanocircuits

► Lecture 3: i) Anderson and Kondo models; ii) DC Transport; iii) Non Fermi liquid extensions

► Lecture 4 (closing the loop): i) Boson-fermion connection in first quantization (using quantum optics concepts); ii) Insights into the Kondo wave-function; iii) Emergent Majorana fermions

: Non-linearities in quantum circuits
Why emphasis on superconducting circuits?

1) Topic is timely with fast progresses:
   - Multiple qubit architecture

2) Connects two topics of the school:
   - Strong link to lectures by Pothier (superconductivity), Splettstoesser (scattering theory), Waintal (finite frequency transport, Luttinger liquids), Alet (many-body wave-functions)

3) Pedagogical: allows to simplify at best tricky concepts
   - No quasiparticle excitation in the Fermi sea: life is easy!
A quick preview on physical systems

Lithographically-defined nano-circuits:

Typical length scale: $L = 100\, \text{nm}$

Complex designs:

- Quantum dot 1
- Surface Acoustic Wave
- Quantum dot 2
- GHz transducer
- Quantum point contact
Scales at play (mesoscopic domain)

Typical length : $L = 100\text{nm}$

Energies :

- Electrostatic : $E = \frac{e^2}{4\pi\epsilon_0 L} \simeq 10^{-22} \text{ J} \simeq 10^{-3}\text{eV}$
- Confinement : $E = \frac{\hbar^2 \pi^2}{meL^2} \simeq 10^{-23} \text{ J} \simeq 10^{-4}\text{eV}$

Temperature : $T = \frac{E}{k_B} \simeq 1\text{K}$ for $E = 10^{-4} \text{ eV}$

Frequency : $f = \frac{E}{2\pi\hbar} \simeq 20\text{GHz}$ for $E = 10^{-4} \text{ eV}$

At cryogenic temperatures ($T = 50\text{mK}$) :

- Single electron charge operations possible
- Quantum confinement takes place (artificial atoms)
- Efficient control by microwave radiation
Plan and purpose of the course

Why do it (besides for the interesting physics)

To build (or not to build?) a quantum computer:

To understand correlated matter in simpler settings:

Non-linearities in quantum circuits
Plan and purpose of the course

What do we mean by non-linearities?

Exemples of non-linear phenomena:

▶ Output not proportional to input
▶ Transport resonances
▶ Frequency conversion

In addition, more fundamentally:

▶ System cannot be described by independent individual entities
  ⇒ forget Landauer-Büttiker

▶ and neither by independent collective entities (on all scales)
  ⇒ forget Fermi liquid theory, Luttinger liquids...
Plan and purpose of the course

A word on methodology in many-body problem

No perfect and unique approach:
Full understanding comes by shining light from various directions

Great variety of methods:
- From simple (but approximate) ones: mean-field, perturbation theory, slave bosons...
- ... To more "exact" (but more technical) ones: many-body Green’s functions and Feynman diagrams, Bethe Ansatz, Conformal Field Theory, Bosonization, advanced numerical algorithms (QMC, NRG, DMRG...)

For this lecture: I will use mostly wave-functions, that are more familiar conceptually, more transparent physically
Plan and purpose of the course

Map of the Hilbert space

Target the physical sector: states accessible upon "reasonable" protocol within decoherence time

How to span the physical sector?
- For low entanglement (1d): Matrix Product States
- Alternative: quantum superposition of "classical" states
- Advantages: input/output friendly, decoherence friendly

Non-linearities in quantum circuits
Superconducting circuitry: waveguides
Josephson junction = high inductance lossless element

Josephson relations: Φ is phase difference across a junction

\[ I = I_c \sin \Phi \simeq I_c \Phi \] (linearized)

\[ V = \frac{\hbar}{2e} \frac{\partial \Phi}{\partial t} \]

\[ \Rightarrow V = \frac{\hbar}{2el_c} \frac{\partial I}{\partial t} = L_J \frac{\partial I}{\partial t} \quad \Rightarrow \quad Z_J(\omega) = iL_J \omega \]

- Josephson inductance density: \( \ell_J \simeq 1 \text{ nH/µm} = 10^4 \ell_{\text{geometric}} \)
- Geometric self-inductance density:
  \[ \ell_{\text{geometric}} = \mu_0/(4\pi) = 10^{-7} \text{ H/m} = 10^{-4} \text{ nH/µm} \]

: Non-linearities in quantum circuits
On the board: Josephson relations
Crash course on circuit-QED for LC-resonator

Classical energy: harmonic oscillator

\[ H = \frac{Q^2}{2C} + \frac{L}{2} I^2 = \frac{Q^2}{2C} + \frac{L}{2} (\dot{Q})^2 \]

Conjugate classical variables: charge/flux

\[ \frac{\partial H}{\partial \dot{Q}} = L \dot{Q} = LI = \phi \]

Quantum regime: \( [\hat{Q}, \hat{\phi}] = i\hbar \) What does it mean?

- Tiny electromagnetic signals generated by charge fluctuations of order \( \delta Q \simeq 2e \) and flux fluctuations of order \( \delta \phi \simeq \hbar/2e \)
- Vacuum reached when \( k_B T \ll \hbar/\sqrt{LC} \) and low loss

Non-linearities in quantum circuits
Josephson junction as a circuit element

Linearized limit (non-linearities for later):

\[ Z(\omega) = \frac{1}{iL_J \omega + jC \omega} = \frac{iL_J \omega}{1 - L_J C \omega^2} \]

Resonance frequency:

\[ f = \frac{1}{2\pi} \sqrt{\frac{1}{L_J C}} \]

\( L_J \approx 1 \text{nH}, \ C \approx 10 \text{pF} \Rightarrow f \approx 1 \text{GHz} \) microwave range

Lumped element limit: Maxwell \rightarrow individual circuit elements

- Associated wavelength \( \lambda = \frac{c}{f} \approx 20 \text{ cm} \)
- \( \lambda \gg \text{juction size (a few } \mu \text{m)} \)
1D Waveguides: optimize coupling to single emitter

Optical domain: requires nanostructuration

- Surface plasmonic modes in metallic nanowires
- Waveguide in photonic crystal
- Tapered optical fiber

Microwave domain: human scale!
On the board: propagation in waveguide
On the board: propagation in waveguide

Telegraph equation for $LC$ waveguide:

\[
I(x, t) = I_1 e^{i\omega [t - \sqrt{\ell c_g} x]} + I_2 e^{i\omega [t + \sqrt{\ell c_g} x]}
\]

\[
V(x, t) = \sqrt{\frac{\ell}{c_g}} \left[ -I_1 e^{i\omega [t - \sqrt{\ell c_g} x]} + I_2 e^{i\omega [t + \sqrt{\ell c_g} x]} \right]
\]

Impedance of waveguide: $Z(\omega) = \sqrt{\ell / c_g} = \sqrt{L / C_g} \approx 3k\Omega$

Crucial point: propagation with velocity $v = 1 / \sqrt{\ell c_g} \approx c / 50$

High inductance $\Rightarrow$ slow light! $\Rightarrow$ stronger light-matter coupling
Josephson junction arrays

Waveguide: a chain of tunnel-coupled superconducting islands

Generic Hamiltonian: valid for $T \ll T_c^{BCS}$

$$H = \frac{1}{2} \sum_{i,j} (2e)^2 n_i [\hat{C}^{-1}]_{ij} n_j - \sum_i E_J \cos(\Phi_i - \Phi_{i+1})$$

$n - \Phi$ are conjugate variables: $[\hat{n}_j, \hat{\Phi}_l] = i\delta_{j,l}$

Non-linearities are controlled by the ratio of charging energy $E_C = (2e)^2 / C$ and Josephson $E_J = \hbar^2 / [(2e)^2 L_J]$ energy
On the board: classical normal modes (general)
On the board: classical normal modes (general)

General result:

\[ \omega_k^2 \text{ are the eigenvalues of symmetric matrix } \hat{C}^{-1/2} \hat{L}^{-1} \hat{C}^{-1/2} \]

valid for any capacitance matrix (even non-uniform waveguides)
On the board: quantum normal modes (waveguide)
Diagonal form:

\[ H_{\text{chain}} = \sum_k \hbar \omega_k a_k^\dagger a_k \]

Spectrum: (same as classically)

\[ \hbar \omega_k = 2 \left| \sin \left( \frac{k}{2} \right) \right| \sqrt{\frac{E_J(2e)^2}{C_g + 4C_J \sin^2(k/2)}} \]

\( k = ap \) dimensionless momentum

Low frequency limit: \( \omega_k \simeq k \sqrt{E_J(2e)^2 / C_g} = k / \sqrt{L_J C_g} \)
we recover slow light

High frequency limit: \( \omega_k \) saturates at \( \omega_P \)
Experimental measurement (@ NEEL)

Finite chain coupled to 50 Ω lines: “giant Fabry-Perot cavity”

\[ T = 20 \text{ mK} \iff f_{\text{thermal}} = \frac{k_B T}{h} = 0.4 \text{ GHz} \ll f_{\text{modes}} : \text{quantum limit} \]

Velocity from dispersion relation: \( v \approx 6 \times 10^6 \text{ m/s} = c/50 \) (slow light)
Tuning the modes

Chain of SQUIDS: $L_J(\phi)$ is flux dependent

Very clean signal: disorder is quite weak

: Non-linearities in quantum circuits
Superconducting circuitry : emitter
Artificial atom engineering

Cooper pair box: gate-tuned Josephson grain

- For \((2e)^2/C \gg E_J\), charge is locked
  - strong phase fluctuations
- Maximize non-linear effects, but not optimal w.r.t. noise
On the board: charge qubit
On the board: charge qubit

At charge degeneracy point:

\[ H = \frac{(2e)^2}{2C} \left[ \hat{N} - N_g \right]^2 - E_J \cos \hat{\Phi} \]

\[ \Rightarrow H = -E_J \frac{\sigma_x}{2} \]

Two-level system:

\[ |g\rangle = \frac{|\uparrow\rangle + |\uparrow\rangle}{\sqrt{2}} \]

\[ |e\rangle = \frac{|\uparrow\rangle - |\uparrow\rangle}{\sqrt{2}} \]
On the board: effective Hamiltonian
On the board: effective Hamiltonian

Spin boson model: [Leggett et al. RMP (1987)]

\[ H = \frac{E_{JQ}}{2} \sigma_x - \sigma_z \sum_k \frac{g_k}{2} (a_k^\dagger + a_k) + \sum_k \omega_k a_k^\dagger a_k \]

Detailed derivation in waveguide QED:

Double dot setup: [LeHur, PRB (2012); Goldstein et al. PRL (2013)]

Cooper pair box: [Snyman, Florens, PRB (2015)]

Spectral density: charge \( \hat{n}_0 \sim \sum_k g_k (a_k^\dagger + a_k) \)

\[ J(\omega) = \text{Im} \langle \hat{n}_0(\omega)\hat{n}_0(-\omega) \rangle = \pi \sum_k g_k^2 \delta(\omega - \omega_k) = 2\pi \alpha \omega e^{-\omega/\omega_P} \]

Interaction strength: effective fine structure constant

\[ \alpha = 2 \left( \frac{C_{cQ}}{C_{cQ} + C_{gQ} + 2C_{JQ}} \right)^2 \frac{Z_{\text{chain}}}{R_Q} \quad \text{with} \quad R_Q = \frac{(2e)^2}{h} \simeq 6k\Omega \]

\( \alpha \) can reach values of order 1
Preliminary experimental results (NEEL)

Inline qubit coupling:

Qubit in regime between CPB (too noisy) and transmon (too linear)

[Nicolas Roch et al., unpublished]
Preliminary experimental results

Inline coupled qubit:

![Graph showing coil current vs. probe frequency with non-linearities in quantum circuits highlighted.](image)
Multimode ultra-strong coupling

Here $\alpha < 0.1$ only, but already a premise of physics beyond standard quantum optics.
On the menu of the next lecture

- Scattering theory with a dynamical degree of freedom
- How to cut a photon into 3 pieces
- Renormalization from many-body entanglement
- Many-body inelastic scattering theory using coherent states