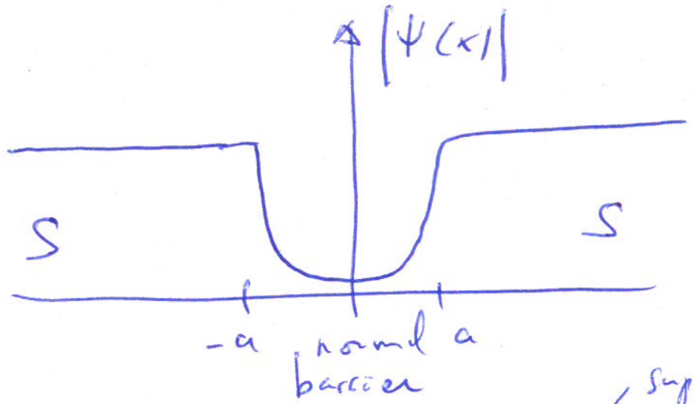
JOSEPHSON RELATIONS

- Super current:

$$j_s = 2e\,\mathrm{Im}\left\{\overbrace{\frac{\hbar}{m}\left[\psi^*\frac{\partial}{\partial x}\psi\right]}^{\text{probability current}}\right\} \quad \text{in 1D}$$



(Landau Ginzburg approach)

$$\left|\begin{aligned} \psi_L(x\to-\infty) &= \overbrace{\sqrt{n}}^{\text{superfluid density}}\, e^{i\varphi_L} \\ \psi_R(x\to+\infty) &= \sqrt{n}\, e^{i\varphi_R} \end{aligned}\right| \quad \text{boundaries}$$

Normal region $\quad -\dfrac{\hbar^2}{2m}\dfrac{\partial^2\psi}{\partial x^2} = \overbrace{(E - V)}^{\text{constant}}\,\psi$

$\Rightarrow \quad \psi(x) = C_1 \,\mathrm{ch}\,\dfrac{x}{\xi} + C_2 \,\mathrm{sh}\,\dfrac{x}{\xi}$ within barrier

evanescent wave

with $\quad \xi = \sqrt{\dfrac{\hbar^2}{2m(V-E)}}$

$\Rightarrow \quad j_s = \dfrac{\hbar e}{m\xi}\,\mathrm{Im}\left(C_1^* C_2\right)$ does not depend on ξ in the normal region ??

But $\quad \psi(-a) = \sqrt{n}\, e^{i\varphi_L}$

$\qquad \psi(a) = \sqrt{n}\, e^{i\varphi_R}$

$$\Rightarrow \begin{cases} C_1 = \dfrac{\sqrt{n}}{2\,ch\left(\frac{a}{\xi}\right)}\left(e^{i\varphi_1} + e^{i\varphi_2}\right) \\ C_2 = -\dfrac{\sqrt{n}}{2\,sh\left(\frac{a}{\xi}\right)}\left(e^{i\varphi_1} - e^{i\varphi_2}\right) \end{cases}$$

$$\Rightarrow j_s = \frac{n\, e\hbar}{m\,\xi\, sh\left(\frac{2a}{\xi}\right)} \sin\left(\varphi_L - \varphi_R\right) = I_c \sin \Phi$$

with $I_c \sim e^{-2a/\xi}$ ok.

. <u>Voltage:</u>

for $I > I_c \Rightarrow$ normal branch with $V_L - V_R \neq 0$

Cooper pairs cannot tunnel $\Rightarrow \Phi$ fluctuates like mad

$$i\hbar \frac{\partial \psi_L}{\partial t} = \frac{-\overset{\swarrow \text{cooper pair charge}}{2e} V}{2}\, \psi_L$$

$$i\hbar \frac{\partial \psi_R}{\partial t} = \frac{2eV}{2}\, \psi_R$$

$$\psi_\alpha = \sqrt{n}\, e^{i\varphi_\alpha(t)} \qquad \alpha = L, R$$

$$\begin{cases} -\hbar \dfrac{d\varphi_L}{dt}\, \psi_L = -\dfrac{2eV}{2}\, \psi_L \\ -\hbar \dfrac{d\varphi_L}{dt}\, \psi_L = \dfrac{2eV}{2}\, \psi_L \end{cases}$$

$$\Rightarrow \boxed{\frac{d\Phi}{dt} = \frac{2e}{\hbar} V} \qquad \text{with } \Phi = \varphi_L - \varphi_R$$

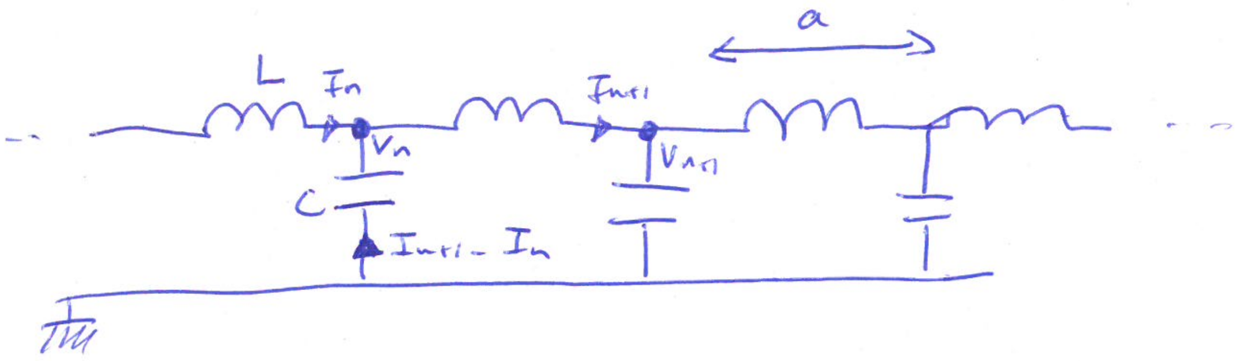
Energy :

$$W = \int_0^t I V \, dt$$

$$W = \int_0^t dt \; I_c \sin\varphi \; \frac{\hbar}{2e} \; \frac{d\varphi}{dt}$$

$$\boxed{W = \frac{I_c \hbar}{2e}\left[1 - \cos\varphi\right]}$$

BOARD : WAVEGUIDE



$$V_{n+1} - V_n = i L \omega I_{n+1}$$

$$\frac{V_{n+1} - V_n}{a} = i \, l \, \omega \, I_{n+1}$$

$$\frac{\partial V(x,t)}{\partial x} = l \frac{\partial}{\partial t} I(x,t) \qquad \text{with } l = \frac{L}{a}$$

$$I_{n+1} - I_n = i C_g \omega V_n$$

$$\frac{\partial I(x,t)}{\partial x} = c_g \frac{\partial V(x,t)}{\partial t} \qquad \text{with } c_g = \frac{C_g}{a}$$

$$\frac{\partial^2 I}{\partial x^2} = c_g \frac{\partial}{\partial t} \frac{\partial V}{\partial x} = l c_g \frac{\partial^2 I}{\partial t^2} \qquad \text{wave equation}$$

$$I(x,t) = I_1 e^{i\omega(t - \sqrt{l c_g}\, x)} + I_2 e^{i\omega(t + \sqrt{l c_g}\, x)}$$

$$V(x,t) = \int^x dx' \; l \frac{\partial I(x',t)}{\partial t}$$

$$V(x,t) = I_1 \frac{-l}{\sqrt{l c_g}} e^{i\omega(t - \sqrt{l c_g}\, x)} + I_2 \frac{l}{\sqrt{l c_g}} e^{i\omega(t + \sqrt{l c_g}\, x)}$$

$$\Rightarrow Z = \sqrt{\frac{l}{c_g}} = \sqrt{\frac{L}{C_g}} \quad \text{and} \quad v_{light} = \frac{1}{\sqrt{l c_g}}$$

Josephson element :

* $L_J = \frac{\hbar}{2e I_c} \simeq \frac{10^{-34}}{10^{-19}\,10^{-12}} = 10^{-34+19+12} = 10^{-3}\ H/m$

$L_J = 1\ nH/\mu m$

* $H = \frac{Q^2}{2C} - \frac{I_c \phi_0}{2\pi} \cos\Phi = \frac{Q^2}{2C} + \frac{1}{2} \underbrace{\frac{I_c \phi_0}{2\pi}}_{E_J} \Phi^2$

$E_J = \frac{I_c \hbar}{2e}$

$I_c = \frac{\hbar}{2e L_J} \Rightarrow E_J = \frac{\hbar^2}{(2e)^2 L_J}$

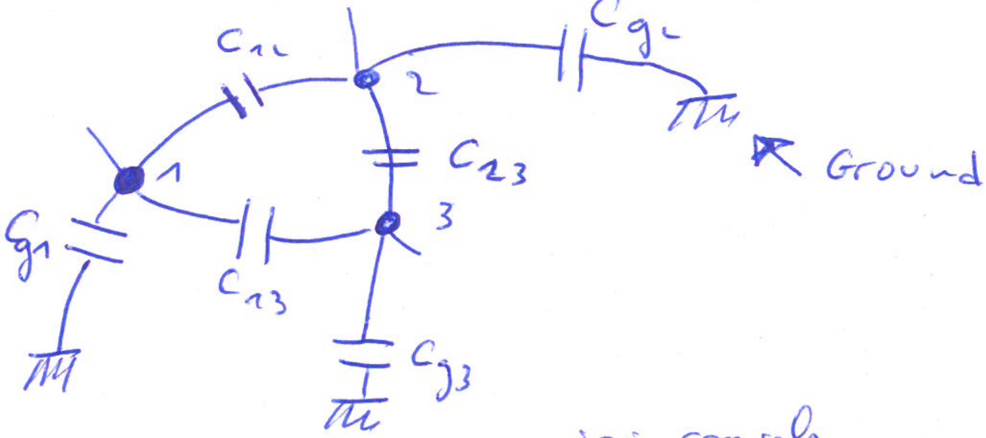
Or $H = \frac{Q^2}{2C} + \frac{1}{2} \underbrace{\frac{\hbar}{2e I_c}}_{L_J} I^2$

* $[\hat{Q}, \hat{\varphi}_J] = i\hbar$

$\underset{\uparrow\ \text{flux}}{\hat{\varphi}_J} = \frac{\hbar}{2e} \underset{\llcorner\ \text{phase}}{\hat{\Phi}} \Rightarrow \left[\frac{\hat{Q}}{2e}, \hat{\Phi}\right] = i$

$\hat{Q} = 2e\hat{N} \Rightarrow \boxed{[\hat{N}, \hat{\Phi}] = i}$

- <u>Capacitance matrix</u>



$j=i$ cancels

$$Q_i = \sum_j C_{ij}(V_i - V_j) + C_{gi} V_i$$

$$Q_i = -\sum_{j \neq i} C_{ij} V_j + \left(\sum_{j \neq i} C_{ij} + C_{gi} \right) V_i$$

$$[\hat{C}]_{ij} = \begin{cases} -C_{ij} & \text{for } i \neq j \\ C_{gi} + \sum_{k \neq i} C_{ik} & \text{for } i = j \end{cases}$$

BOARD : CLASSICAL NORMAL MODES

CNM 1

$$H = \frac{(2e)^2}{2} \sum_{ij} (C^{-1})_{ij}\, n_i n_j + V(\Phi)$$

$$Q_i = \sum_j \hat{C}_{ij}\, V_j$$

$$Q_i = \sum_j \hat{C}_{ij}\, \frac{\hbar}{2e} \frac{\partial \Phi_j}{\partial t}$$

$$n_i = \sum_j \hat{C}_{ij}\, \frac{\hbar}{(2e)^2}\, \dot{\Phi}_j$$

$$H = \frac{1}{2} \frac{\hbar^2}{(2e)^2} \sum_{ij} C_{ij}\, \dot{\Phi}_i \dot{\Phi}_j + V(\Phi)$$

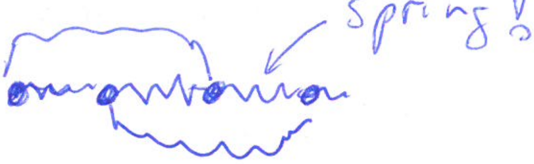
$$\Phi_i = \sum_l \left[\hat{C}^{-1/2}\right]_{il} x_l \frac{2e}{\hbar}$$

$$H = \frac{1}{2} \sum_i (\dot{x}_i)^2 + V\left(\hat{C}^{-1/2}\, \vec{x}\, \frac{2e}{\hbar}\right)$$

$$V(\Phi) = \frac{E_J}{2} \sum_i (\Phi_i - \Phi_{i+1})^2$$

$$= \frac{\hbar^2}{(2e)^2 L} \sum_i (\Phi_i - \Phi_{i+1})^2$$

$\Longrightarrow$ harmonic chain with arbitrary couplings



$$V[\Phi] = \frac{\hbar^2}{2e^2} \sum_{ij} \Phi_i \hat{L}^{-1}_{ij} \phi_j$$

$$V[\varphi] = \frac{1}{2} \sum_{ij} x_i \left[\hat{C}^{-1/2} \hat{L}^{-1} \hat{C}^{-1/2}\right]_{ij} x_j$$

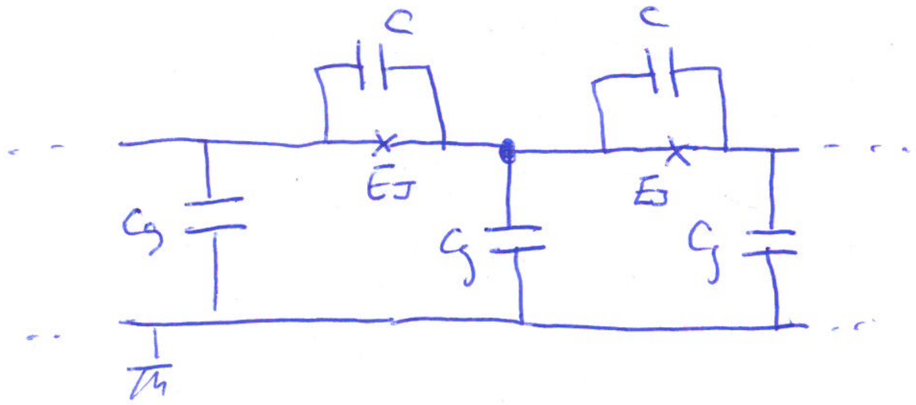
Eq. of motion $\quad \ddot{x}_i = -\frac{\partial V}{\partial x_i}$

$$\ddot{\vec{x}} = -C^{-1/2} \hat{L}^{-1} \hat{C}^{-1/2} \vec{x}$$

$\Rightarrow$ $\boxed{\omega_k^2 \text{ eigenvalue of } \hat{C}^{-1/2} \hat{L}^{-1} \hat{C}^{-1/2}}$

This applies to any $\hat{C}$ matrix, not necessarily translat° invariant (provided one can linearize the Josephson energy).

BOARD : JOSEPHSON WAVEGUIDE



Assume infinite waveguide

Capacitance matrix:

$$\hat{C} = \begin{bmatrix} \ddots & \ddots & -C & & \\ -C & C_g+C & -C & & (0) \\ (0) & -C & C_g+C & \ddots & \\ & & \ddots & \ddots & \end{bmatrix} \Rightarrow [\hat{C}]_{\ell j} = [\hat{C}](\ell - j)$$

Fourier modes:

$$n_j = \int_{-\pi}^{\pi} \frac{dk}{2\pi}\, n_k e^{ikj} \qquad n_j \text{ real} \Rightarrow n_k^* = n_{-k}$$

$$\Phi_j = \int_{-\pi}^{\pi} \frac{dk}{2\pi}\, \Phi_k e^{-ikj}$$

sign change ?

This is inverted by:

$$n_k = \sum_{j=-\infty}^{+\infty} n_j e^{-ikj}$$

as $n_j = \int_{-\pi}^{\pi} \frac{dk}{2\pi} e^{ikj} \sum_{\ell=-\infty}^{+\infty} n_\ell e^{-ik\ell}$

$$n_j = \sum_\ell n_\ell \underbrace{\int_{-\pi}^{\pi} \frac{dk}{2\pi} e^{ik(j-\ell)}}_{\delta_{j\ell}} = n_j \text{ ok.}$$

Hamiltonian:

$$H = \frac{(2e)^2}{2} \sum_{lj} (C^{-1})_{lj}\, n_l n_j + \frac{E_J}{2} \sum_l (\phi_l - \phi_{l+1})^2$$

$$H = \frac{(2e)^2}{2} \int_{-\pi}^{\pi} \frac{dk}{2\pi} \int_{-\pi}^{\pi} \frac{dk'}{2\pi}\, n_k n_{k'} \sum_{lj} [\hat{C}^{-1}]_{l-j}\, e^{ikl + ik'j}$$

$$+ \frac{E_J}{2} \int \frac{dk}{2\pi} \int \frac{dk'}{2\pi}\, \phi_k \phi_{k'} \sum_l \left[e^{-i(k+k')l} + 2\, e^{-ikl - ik'(l+1)} + e^{-i(k+k')(l+1)} \right]$$

$$\begin{cases} l - j = m \\ l + j = n \end{cases}$$

$$H = \frac{(2e)^2}{2} \int_{-\pi}^{\pi} \frac{dk}{2\pi} \int_{-\pi}^{\pi} \frac{dk'}{2\pi}\, n_k n_{k'} \sum_{m=-\infty}^{+\infty} [\hat{C}^{-1}]_m\, e^{ik\frac{m+n}{2} + ik'\frac{n-m}{2}}$$

$$+ \frac{E_J}{2} \int_{-\pi}^{\pi} \frac{dk}{2\pi} \int_{-\pi}^{\pi} \frac{dk'}{2\pi}\, \phi_k \phi_{k'}\, 2\pi\delta(k+k') \left[1 - 2\, e^{-ik'} + 1 \right]$$

$$H = \frac{(2e)^2}{2} \int_{-\pi}^{\pi} \frac{dk}{2\pi} \int_{-\pi}^{\pi} \frac{dk'}{2\pi}\, n_k n_{k'} \sum_m [\hat{C}^{-1}]_m\, e^{i(k-k')\frac{m}{2}} \overbrace{\sum_n e^{i\frac{k+k'}{2}n}}^{2\delta(k+k')}$$

$$+ \frac{E_J}{2} \int \frac{dk}{2\pi}\, \phi_k \phi_{-k} \left(2 - 2\, e^{-ik} \right)$$

$$H = \frac{(2e)^2}{2} \int_0^{\pi} \frac{dk}{2\pi}\, n_k n_{-k} \frac{1}{C_k}$$

$$+ 2E_J \int_0^{\pi} \frac{dk}{2\pi}\, \phi_k \phi_{-k} \left(1 - e^{-ik} \right)$$

$\frac{e^{ik} + e^{-ik}}{2}$ by $k \to -k$ sym

$$C_m = (C_J + C)\,\delta_{m,0} - C\,\delta_{m,1} - C\,\delta_{m,-1}$$

$$C_k = \sum_m C_m\, e^{-ikm}$$

$$C_k = (C_J + C) - C\, e^{-ik} - C\, e^{ik}$$

WRONG

$$C_k = (C_J + C) - 2C\cos(k)$$

$$H = \frac{(2e)^2}{2} \int_0^{\pi} \frac{dk}{2\pi}\, n_k\, n_{-k}\, \frac{2}{C_J + C - 2C\cos(k)}$$

$$+ E_J \int_0^{\pi} \frac{dk}{2\pi}\, \phi_k\, \phi_{-k}\, \left[1 - \cos(k)\right]$$

$$\boxed{H = \frac{1}{2}\int_0^{\pi} \frac{dk}{2\pi} \left\{ \frac{(2e)^2}{C_J + 2C - 2C\cos(k)}\, n_k n_{-k} + E_J\left[1-\cos(k)\right]\phi_k \phi_{-k} \right\}}$$

with $[n_k, \phi_{k'}] = \sum_{j\ell} \underbrace{[n_j, \phi_\ell]}_{i\,\delta_{j\ell}} e^{ikj - ik'\ell} = i \underbrace{\sum_j e^{i(k-k')j}}_{\delta(k+k')}$

$[n_k, \phi_{k'}] = i\,\delta(k-k')$ ok ?

$$n_k = \theta_k \frac{b_k + b_k^+}{\sqrt{2}}$$

$$\phi_k = \xi_k\, i\, \frac{b_k - b_k^+}{\sqrt{2}}$$

$n_k^+ = n_{-k} \Rightarrow b_k = b_{-k}$

$\phi_k^+ = \phi_{-k} \Rightarrow b_k = b_{-k}$ also

JW ④

$$[n_k, \phi_{k'}] = \Theta_k \xi_{k'} \frac{i}{2} [b_k + b_k^+, b_{k'} - b_{k'}^+]$$

$$= \Theta_k \xi_{k'} \frac{i}{2} [-\delta(k-k') - \delta(k-k')]$$

$$= -\Theta_k \xi_k \, i \, \delta(k-k')$$

$$\Rightarrow \xi_k = -\frac{1}{\Theta_k}$$

$$H = \int \frac{dk}{2\pi} \left\{ \frac{(2e)^2}{C_g + C - 2C\cos k} \frac{\Theta_k^2}{2} (b_k + b_k^+)(b_k + b_k^+) \right.$$

$$\left. - E_J (1 - \cos k) \frac{1}{2\Theta_k^2} (b_k - b_k^+)(b_k - b_k^+) \right\}$$

$$\frac{(2e)^2}{C_g + C - 2C\cos k} \frac{\Theta_k^2}{2} + E_J (1 - \cos k) \frac{1}{2\Theta_k^2} = 0$$

$$\Theta_k^2 = \sqrt{E_J (1 - \cos k) \cdot \frac{(C_g + C - 2C\cos k)}{(2e)^2}}$$

$$H = \int dk \frac{(2e)^2}{C_g + C - 2C\cos k} \sqrt{E_J (1 - \cos k) \frac{C_g + C - 2C\cos k}{(2e)^2}} \, b_k^+ b_k$$

$$+ \frac{E_J (1 - \cos k)}{\sqrt{E_J (1 - \cos k) \frac{(C_g + C - 2C\cos k)}{(2e)^2}}} \, b_k^+ b_k$$

$$H = \int \frac{dk}{2\pi}\, \hbar\omega_k \; b^\dagger_k b_k$$

$$\hbar\omega_k = 2\sqrt{E_J(1-\cos k)}\;(2e) \Big/ \sqrt{C_g + C - 2C\cos k}$$

$$\boxed{\hbar^2\omega_k^2 = \frac{4(2e)^2 E_J \sin^2\left(\frac{k}{2}\right)}{C_g + 4C\sin^2\left(\frac{k}{2}\right)}}$$

dispersion relation

- Physics:

$\omega \to 0$  telegraph

$\omega \to \infty$  short circuit

- philosophy:

We have seen three equivalent descriptions of the waveguide.

1) for the electrical engineer
2) for the classical mechanist
3) for the quantum mechanist

COOPER PAIR BOX

$$H = \frac{(2e)^2}{2C} (\hat{N} - N_g)^2 - E_J \cos\hat{\phi}$$

with $[\hat{N}, \hat{\phi}] = i$

no ħ here since $[\hat{N}, \hat{\phi}] = i$

$$\Rightarrow \quad H = \underbrace{+ \frac{(2e)^2}{2C} \left(i \frac{\partial}{\partial \phi} - N_g \right)^2}_{H_0} - E_J \cos\phi$$

$$H_0 \Psi(\phi) = E \Psi(\phi)$$

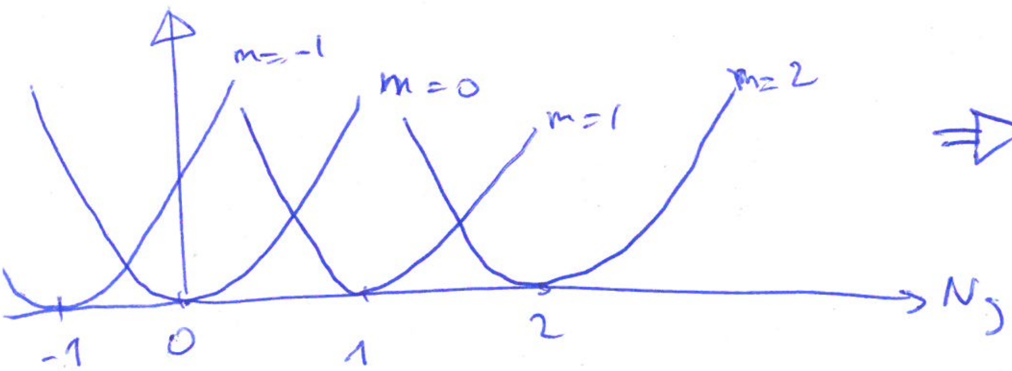
$$\frac{(2e)^2}{2C} \left(- \frac{\partial^2}{\partial \phi^2} - 2 i N_g \frac{\partial}{\partial \phi} + N_g^2 \right) \Psi = E \Psi$$

Look for "plane wave" with $\Psi(0) = \Psi(2\pi)$

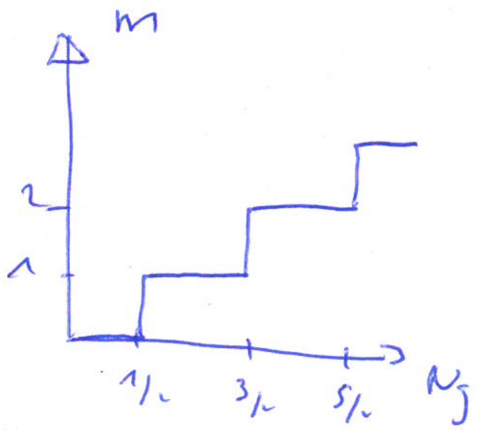
$$\Rightarrow \Psi(\phi) = e^{im\phi}$$

$$\Rightarrow \frac{(2e)^2}{2C} (m^2 - 2 N_g m + N_g^2) \Psi = E \Psi$$

$$\Rightarrow E_m = \frac{(2e)^2}{2C} (m - N_g)^2$$



$\Rightarrow$



Coulomb blockade

Charge state ψ_n has $|\psi(\phi)|^2 = 1 \quad \forall \phi$

$\Rightarrow$ maximal phase fluctuations of phase.

· <u>Charge qbit:</u>

Sit at degeneracy point between $n=0$ and $n=1$

$$|\psi_0\rangle = |\uparrow\rangle$$

$$|\psi_1\rangle = |\downarrow\rangle$$

constant drops

Let's consider $H = \frac{(2e)^2}{2C}\left(\frac{1}{2}\right) - E_J \cos\phi$

$$\cos\phi = \frac{e^{i\phi} + e^{-i\phi}}{2}$$

$$\langle\phi| \cos\phi |\psi_n\rangle = \frac{e^{i\phi} + e^{-i\phi}}{2} \psi_n(\phi)$$

$$= \frac{1}{2}\left(\psi_{n+1} + \psi_{n-1}\right)$$

$$\Rightarrow \cos\phi = \frac{1}{2}\sum_{m=-\infty}^{+\infty} |m\rangle\langle m+1| + |m\rangle\langle m-1|$$

for $E_C \gg E_J$, forget $n \neq 0, 1$

$$\Rightarrow H = -\frac{E_J}{2}\left(|\uparrow\rangle\langle\downarrow| + |\downarrow\rangle\langle\uparrow|\right)$$

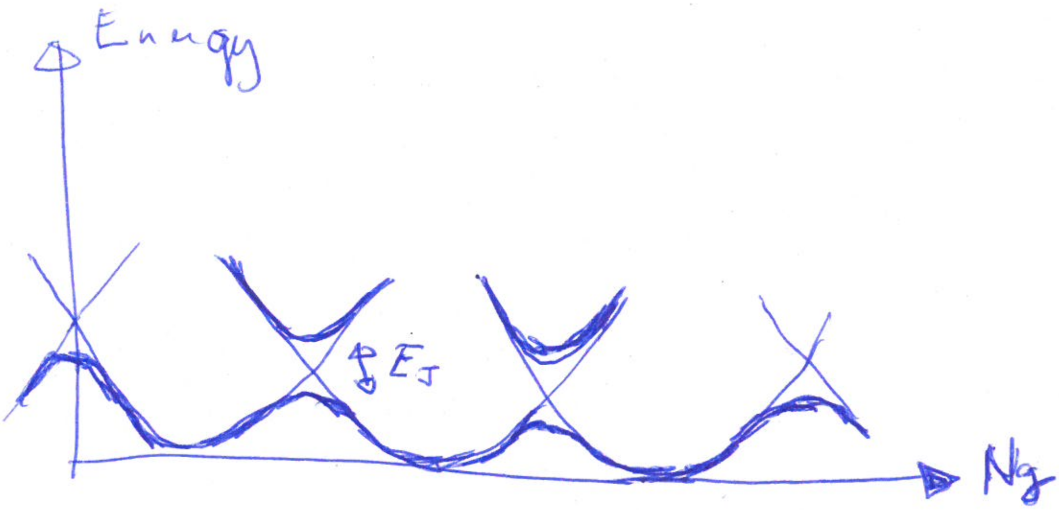
$$\boxed{H = -E_J \frac{\sigma_x}{2}}$$

→ anticrossing of amplitude E_J

with eigenstates

$$|g\rangle = \frac{|\uparrow\rangle + |\downarrow\rangle}{\sqrt{2}}$$

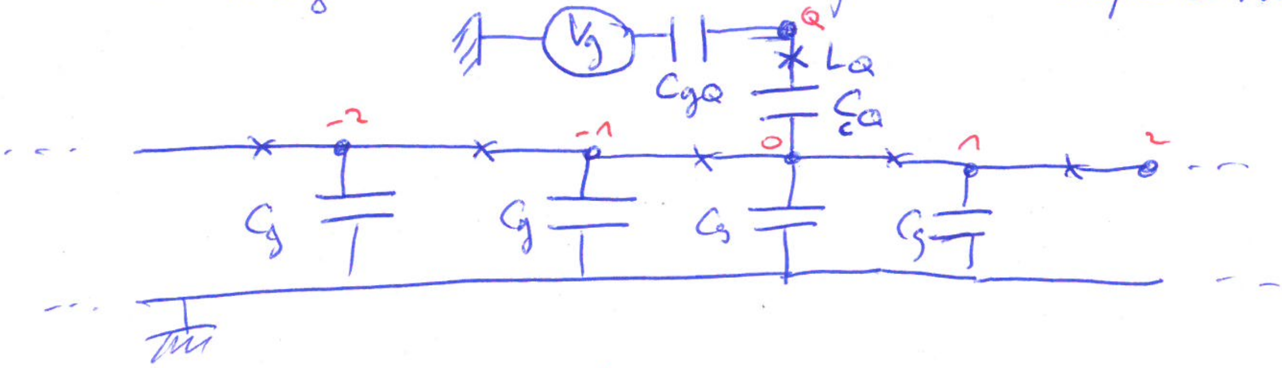
$$|e\rangle = \frac{|\uparrow\rangle - |\downarrow\rangle}{\sqrt{2}}$$



QBIT + CHAIN

· <u>Design:</u> side coupling capacitively

We neglect here all junct⁼ capacitances



$$\hat{C} = \begin{array}{c} \begin{array}{cccccc} \quad & -1 & 0 & Q & 1 & \quad \end{array} \\ \begin{bmatrix} \ddots & & & & \\ & C_g & & & \\ & & C_g + C_{cQ} & -C_{cQ} & \\ & & -C_{cQ} & C_{gQ} & \\ & & & & C_g \\ & & & & & \ddots \end{bmatrix} \end{array}$$

$$\hat{C}^{-1} = \begin{bmatrix} \ddots & & & & \\ & 1/C_g & & & \\ & & \frac{C_{gQ}}{D} & \frac{C_{cQ}}{D} & \\ & & \frac{C_{cQ}}{D} & \frac{C_g + C_{cQ}}{D} & \\ & & & & 1/C_g \\ & & & & & \ddots \end{bmatrix}$$

with $D = (C_g + C_{cQ}) C_{gQ} + (C_{cQ})^2$

$$\Rightarrow H_{coupling} = (2e)^2 \frac{C_{cQ}}{D} \hat{n}_0 \hat{n}_Q$$

$$B_{ur}(2e)\frac{c_a}{D}\hat{n}_0 = \sum_k g_k \left(b_k^\dagger + b_k\right)$$

and $\hat{n}_e = \frac{\sigma_z}{2}$

$$\Rightarrow \boxed{H = -\frac{E_J}{2}\sigma_x + \frac{\sigma_z}{2}\sum_k g_k\left(b_k^\dagger + b_k\right) + \sum_k \omega_k a_k^\dagger a_k}$$

Spin boson model = quantum optics

- RWA: in eigen basis

$\sigma^+ = |+\rangle\langle -|$

$-\frac{E_J}{2}\sigma_x \longrightarrow +\frac{E_J}{2}\tau_z$ $\qquad |g\rangle = |-\rangle_{\tau_z}$

$\sigma_z \longrightarrow \tau_x$ $\qquad |e\rangle = |+\rangle_{\tau_z}$

$$H = \frac{E_J}{2}\tau_z + \frac{\tau_x}{2}\sum_k g_k\left(b_k^\dagger + b_k\right) + \ldots$$

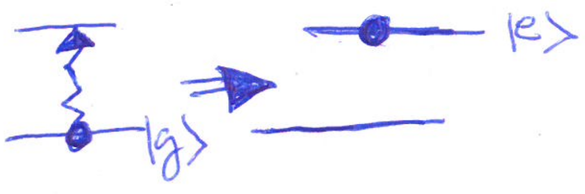
$$H = \frac{E_J}{2}\tau_z + \frac{\tau_+ + \tau_-}{2}\sum_k g_k\left(b_k^\dagger + b_k\right)$$

$$= \frac{E_J}{2}\tau_z + \sum_k g_k\left[\tau_+ b_k + \tau_- b_k^\dagger\right]$$

$$+ \sum_k g_k\left[\tau_+ b_k^\dagger + \tau_- b_k\right]$$

[redacted]

RWA process



Full expression:

$$\omega_k = 2\sin\frac{k}{2}\sqrt{\frac{(2e)^2 E_J}{C_g + 4C\sin^2\left(\frac{k}{2}\right)}}$$

$$g_k = \frac{1}{\sqrt{2}}\frac{C_{cQ}}{C_{gQ} + C_{cQ}}\frac{\omega_k}{\sin\left(\frac{k}{2}\right)}\sqrt{\frac{\omega_k}{2\pi E_J}}\cos\left(\frac{k}{2}\right)$$

Low k: $g_k \propto \frac{k}{k}\sqrt{k} \propto k$

$\omega_k \propto k$

High k: ω_k saturates $\Rightarrow$ high DoS.

But $g_k \xrightarrow[k\to\pi]{} 0$ quadratically.

Spectral density:

this is the density associated to

$$\hat{n}_0 = \sum_k g_k \left(b_k^\dagger + b_k\right)$$

$$J(\omega) = \mathrm{Im}\left\langle \hat{n}_0(\omega)\,\hat{n}_0(-\omega)\right\rangle = \pi\sum_k g_k^2\,\delta(\omega - \omega_k)$$

$$\boxed{J(\omega) = 2\pi\alpha\,\omega\, e^{-\omega/\omega_p} \quad \text{good model}}$$