Anderson localization of cold atoms in a disordered speckle potential

Introduction : Atomic ultra-cold matter waves may interfere, as exemplified by the development of atomic interferometry during the last 20 years. This problem studies how to create a specifically tailored disorder acting on the external motion of cold atoms and how such a disorder may influence this motion, with emphasis on Anderson localization.

In the first part, you have to calculate the statistical properties of a "speckle" pattern created by a coherent laser beam transmitted through a diffusive glass plate. In the second part, you have to use these statistical properties to study the dynamics of a cold atomic gas placed in such a speckle pattern. **The two parts are essentially independent.**

Part 1: Statistical properties of a speckle pattern

We consider the situation depicted in figure 1, where a monochromatic laser beam (approximated by a plane wave with wave-vector k_0 along the z-axis) is diffracted by a square aperture (side L) in the plane $z = 0$. We will assume that all light sources have the same polarization, so that we can forget about light polarization and describe the electromagnetic field by a single complex variable $\mathcal{E}$.
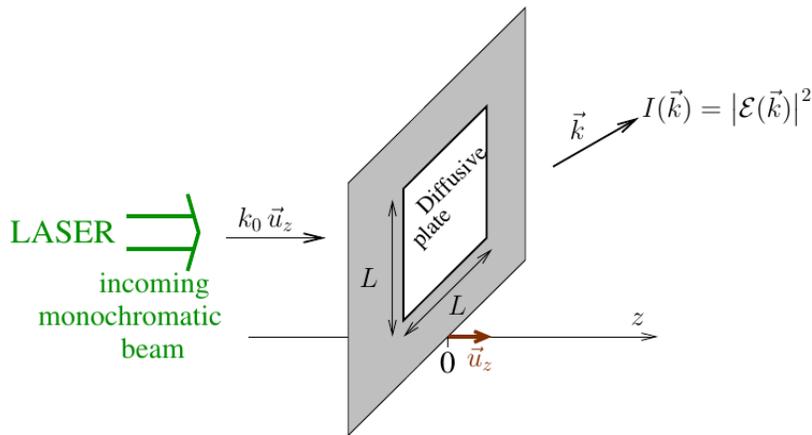

Figure 1: *A monochromatic laser beam along the z-axis is sent on a square aperture of size L in the z = 0 plane. We consider the diffracted field at large distance in the $\vec{k}$ direction, close to the initial direction (paraxial approximation).*

Fraunhofer regime.– In a direction close to the z−axis (in the paraxial approximation) one can use the Fraunhofer formula, relating the amplitude of the field at the level of the aperture ($z = 0$), denoted $A(x, y)$, to the amplitude of the field diffracted in the $\vec{k}$

direction (with k_x, $k_y \ll k_0$ and $k_z \approx k_0$) :

$$\mathcal{E}(\vec{k}) = \int_{-L/2}^{L/2} \mathrm{d}x \int_{-L/2}^{L/2} \mathrm{d}y\, A(x, y)\, \exp\left[\mathrm{i}(k_x x + k_y y)\right] . \tag{1}$$

A. Warm up : diffraction by a square aperture.– We want to study the amplitude of the wave diffracted by a square aperture, **in the absence of any diffusive plate**, i.e. $A(x, y) = \mathcal{A}$ is constant. Compute $\mathcal{E}(\vec{k})$ and the corresponding intensity $I(\vec{k}) = |\mathcal{E}(\vec{k})|^2$. It may be useful to introduce the so-called sinc function : $\mathrm{sinc}(x) \overset{\text{def}}{=} \sin x / x$.

B. Diffraction by a diffusive plate.– We now add on the square aperture a diffusive glass plate whose effect is to modify the *phase* of $A(x, y)$ differently at each position (x, y). We assume that the phase (mod. 2π) is a random variable, homogenously distributed in $[0, 2\pi[$, and which is moreover uncorrelated between different positions.

1/ Distribution of the electric field.– Justify, with these hypotheses, the following average values:

$$
\begin{aligned}
\langle A(\vec{r}) \rangle &= 0 \\
\langle A(\vec{r}) A(\vec{r}') \rangle &= 0 \\
\langle A(\vec{r}) A^*(\vec{r}') \rangle &= \mathcal{A}^2\, \eta^2\, \delta_\eta(\vec{r} - \vec{r}')
\end{aligned}
$$

where δ_η is a (normalized, i.e. integral=1) 2D Dirac "distribution" of width η in each direction (x and y).

   Explain why the wave $\mathcal{E}(\vec{k})$ diffracted in direction $\vec{k}$ is a complex random variable with a **Gaussian** distribution with zero mean, $\langle \mathcal{E}(\vec{k}) \rangle = 0$.

Hint : you can discretize expression (1) as $\mathcal{E}(\vec{k}) = \eta^2 \sum_{n=1}^{N} A(\vec{r}_n)\, \mathrm{e}^{\mathrm{i}\vec{k}\cdot\vec{r}_n}$, where $\vec{r}_n$ are $N = (L/\eta)^2$ discrete and regular positions on the square (hence $\langle A(\vec{r}_n) A^*(\vec{r}_m) \rangle = \mathcal{A}^2\, \delta_{n,m}$), and use a well-known theorem of probability theory.

Justify that $\mathcal{X} = \mathrm{Re}(\mathcal{E})$ and $\mathcal{Y} = \mathrm{Im}(\mathcal{E})$ have the same variance and are uncorrelated.

2/ Rayleigh law.– Deduce that the intensity $I(\vec{k}) = |\mathcal{E}(\vec{k})|^2$ is distributed according to an exponential law :

$$P(I) = \frac{\exp\left(-I/I_0\right)}{I_0} \qquad \text{for } I \geqslant 0 \tag{2}$$

where I_0 is the average intensity. What is the most probable intensity? Compute the variance of the intensity.

3/ Field correlations.– We are interested in the correlation function of the diffracted amplitude and intensity in different directions $\vec{k}$ and $\vec{k}'$.

a) Show that $\langle \mathcal{E}(\vec{k}) \mathcal{E}(\vec{k}') \rangle = 0$. Compute the correlation function $\langle \mathcal{E}(\vec{k}) \mathcal{E}^*(\vec{k}') \rangle$ and express it as a function of $\Delta \vec{k} = \vec{k} - \vec{k}'$, L and I_0.

Hint : use the hint of question **1/**.

b) Using the Wick's theorem, show that:

$$\langle I(\vec{k}) I(\vec{k}') \rangle = I_0^2 \; + \; \left| \langle \mathcal{E}(\vec{k}) \mathcal{E}^*(\vec{k}') \rangle \right|^2 \tag{3}$$

c) Show that one recovers the variance of the intensity previously computed. Extend the argument for the higher moments $\langle I(\vec{k})^n \rangle$. Compare with the Rayleigh law found above.

4/ Spatial speckle pattern.– The setting under study can be used in order to create a *spatially* (instead of angularly) correlated laser intensity.

a) Explain briefly how, using a lens of focal length f, it is possible to create such a potential in a convenient plane. If we denote by $\vec{r} = (x, y)$ the coordinate in this plane, how a variation of direction $\Delta\vec{k}$ is related to a variation in position $\Delta\vec{r}$ in this plane (we recall that k_x, $k_y \ll k_z \approx k_0$)?

b) Suppose that one puts a gas of non-interacting cold atoms in this plane, and that the laser frequency $\omega = c||\vec{k}||$ is quasi-resonant with an atomic resonance transition at frequency ω_0. Briefly explain the relevant mechanisms of the atom-light interaction and why, if the detuning $\delta = \omega - \omega_0$ is large compared to the width Γ of the atomic resonance, the speckle pattern acts as an effective "optical" potential V which depends on the position $\vec{r}$ of the center of mass of the atom. Show that the correlation function of the potential is given by :

$$\langle V(\vec{r})V(\vec{r}+\Delta\vec{r})\rangle = V_0^2 \left[1 + \mathrm{sinc}^2\left(\frac{\Delta x}{\sigma}\right) \mathrm{sinc}^2\left(\frac{\Delta y}{\sigma}\right) \right] \tag{4}$$

where σ is the correlation length of the speckle. Express σ as a function of the laser wavelength $\lambda = 2\pi/k_0$, L and f. Considering the approximations used in the calculation, what is the order of magnitude of the shortest correlation length achievable?

C. Diffraction by a Gaussian aperture.– We now consider the case where there is no aperture, but the incoming laser beam has a Gaussian distribution of amplitude in the $z = 0$ plane : $|A(x,y)| = \mathcal{A} \exp\left(-\frac{x^2+y^2}{2r_0^2}\right)$.

1/ By following the same lines as in question **B.3/**, deduce the field correlation $\langle \mathcal{E}(\vec{k})\mathcal{E}^*(\vec{k}')\rangle$ in this case.

2/ What would be the corresponding correlations replacing Eq. (4)?

Part 2: Cold atoms in a 1D speckle potential

We want to study the dynamics of **one atom** of mass m exposed to a spatially disordered potential $V(\vec{r})$ whose correlation function is given by eq. (4). We assume that some external constraint forces the atom to move only along the x-axis, so that the entire **dynamics is one-dimensional** with Hamiltonian :

$$H = H_0 + V(x) = \frac{p_x^2}{2m} + V(x) \tag{5}$$

with $\langle V(x)\rangle = V_0$ and

$$\langle V(x)V(x+\Delta x)\rangle = V_0^2 \left[1 + \mathrm{sinc}^2\left(\frac{\Delta x}{\sigma}\right) \right] \tag{6}$$

A. Structure of the Green functions

1/ *a*) We introduce the retarded Green function $G_0(E) = 1/(E - H_0 + i0^+)$ (we omit the "R" index to lighten). Express the Green function in k-space, $\widetilde{G}_0(k, k'; E) \stackrel{\text{def}}{=} \langle \phi_k | G_0(E) | \phi_{k'} \rangle$,

Hint : $\phi_k(x) = \langle x | \phi_k \rangle = (1/\sqrt{L})e^{ikx}$ denotes a plane wave in finite volume L, with quantized wave vectors $k = 2n\pi/L$ for $n \in \mathbb{Z}$, and orthonormalisation $\langle \phi_k | \phi_{k'} \rangle = \delta_{k,k'}$ (thus $\sum_k \leftrightarrow \frac{L}{2\pi} \int dk$).

b) Recover the corresponding expression in x-space,

$$G_0(x, x'; E) = \frac{m}{i\,\hbar^2 k_E} \exp(ik_E|x - x'|) \qquad \text{where } k_E \stackrel{\text{def}}{=} \sqrt{2mE}/\hbar. \tag{7}$$

2/ Recall the relation between the Green function $G(E)$ (for Hamiltonian H), $G_0(E)$ and V. Briefly explain the notion of average Green function $\overline{G}(E)$. The self-energy $\Sigma(E)$ is the operator such that :

$$\overline{G}(E) = \frac{1}{G_0^{-1}(E) - \Sigma(E)} \tag{8}$$

Recall the Dyson equation satisfied by $\Sigma(E)$. Give a physical argument to explain that the self energy is diagonal in the momentum representation : $\langle \phi_k | \Sigma(E) | \phi_{k'} \rangle = \Sigma(k; E)\,\delta_{k,k'}$.

3/ We consider the weak disorder regime $|\Sigma(k; E)| \ll |E|$. We assume that the self-energy is a *smooth* function of k and E. What is the physical interpretation of $\text{Re}\left[\Sigma(k; E)\right]$ and $\text{Im}\left[\Sigma(k; E)\right]$? Show that the average Green function in x-space can be written as :

$$\overline{G}(x, x'; E) \approx G_0(x, x'; E) \exp\left(-\frac{|x - x'|}{2\ell(E)}\right), \tag{9}$$

where $\ell(E)$ can be expressed in terms of $\Sigma(k_E; E)$. What is the physical meaning of $\ell(E)$?

B. Born approximation.$-$ It is possible to perform a diagrammatic expansion of $\Sigma(k; E)$ in powers of the disorder strength V_0. Up to second order, it is :

$$\Sigma(k; E) \simeq \quad \xrightarrow{k} \bullet \xrightarrow{k} \quad + \quad \xrightarrow{k} \overset{k - k'}{\underset{k'}{\triangle}} \xrightarrow{k} \tag{10}$$

1/ Explain the meaning of the various parts of these diagrams.

2/ Compute $\Sigma(k; E)$ at first order in V_0. Interpret the result.

3/ Write the integral expression for the second order term (the so-called "Born approximation"). Show that the imaginary part of the "on-shell" self-energy (that is for $k_E = \sqrt{2mE}/\hbar$) is given by :

$$\text{Im}\,\Sigma(k_E; E) \simeq -\frac{mV_0^2}{2\hbar^2 k_E}[\mathcal{C}(0) + \mathcal{C}(2k_E)] \tag{11}$$

where $\mathcal{C}(k)$ is the Fourier transform of the correlation function of the potential

$$V_0^2\,\mathcal{C}(k) \stackrel{\text{def}}{=} \int_{-\infty}^{+\infty} dy \left[\langle V(x_0)V(x_0 + y)\rangle - \langle V(x_0)\rangle^2\right] e^{-iky} \tag{12}$$

Deduce the expression for $1/k_E \ell(E)$ where $\ell(E)$ is the scattering mean free path. Which physical phenomena are respectively associated with $\mathcal{C}(0)$ and $\mathcal{C}(2k_E)$?

C. Localization length

1/ Preliminary : a Fourier transform.– By a simple argument (no need of an explicit calculation), justify that $\mathcal{C}(k)$ deduced from (6) is a triangle function $\propto (1 - |k|/a) \, \theta_{\mathrm{H}}(1 - |k|/a)$, where θ_{H} is the Heaviside function, and give the width a.

Hint : Fourier transform of the $\mathrm{sinc}(x)$ function is the "door" function $\pi \, \theta_{\mathrm{H}}(1 - |k|)$.

2/ *a*) Explain why, when placed inside the 1D speckle potential, a cold atom is expected to be Anderson localized. When scattering by the disorder is isotropic ($\mathcal{C}(k)$ is independent on k), recall (without justification) how the localization length ξ_{loc} is related to the scattering mean free path ℓ.

b) When scattering is not isotropic (i.e. forward and backward scattering have different probabilities), the simple relation discussed in **2.*a*** must be understood as a relation between the localization length ξ_{loc} and the *transport* mean free path ℓ_{tr}. The transport mean free path ℓ_{tr} is obtained by changing $\mathcal{C}(0) \to \mathcal{C}(2k)$ in the expression of the mean free path ℓ (we now simplify the notation as $k_E \to k$ and omit the E dependences).

c) Show that the inverse of the localization length is given by :

$$\frac{1}{\xi_{\mathrm{loc}}} \simeq \frac{\pi \, m^2 \, V_0^2 \, \sigma \, (1 - k\sigma) \, \theta_{\mathrm{H}}(1 - k\sigma)}{\hbar^4 k^2} \ . \tag{13}$$

d) Fig. 2 shows the numerically computed inverse localization length $1/\xi_{\mathrm{loc}}^{(\mathrm{num})}$ *versus* $k\sigma$ [1]. Comment this figure. Especially, what is the problem around $k\sigma = 1$? How should the above analysis be adapted in order to explain the numerics for $k\sigma > 1$?
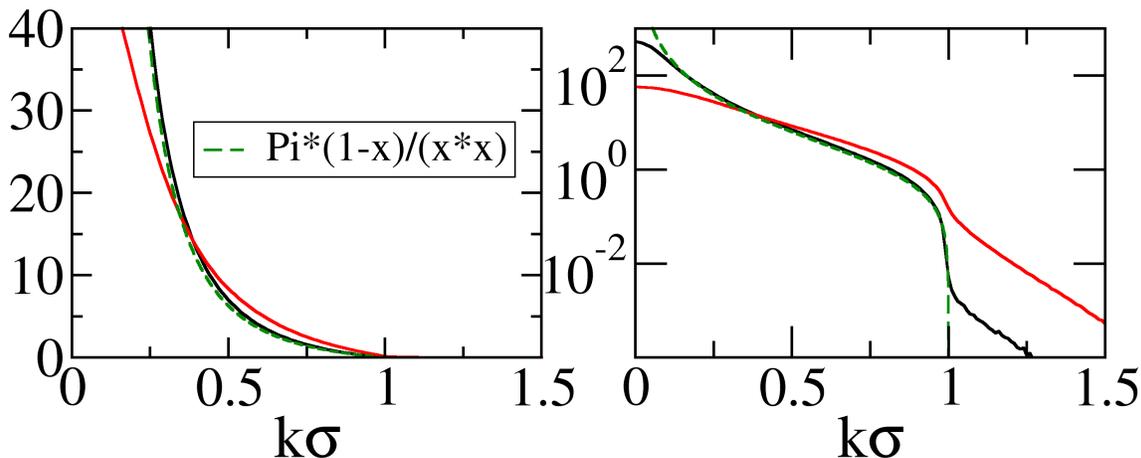

Figure 2: *Numerically computed inverse localization length for a 1D particle in a disordered "speckle" potential, versus. the parameter $k\sigma$. The dimensionless plotted quantity is $(\sigma/\xi_{\mathrm{loc}}) \, (\hbar^4/m^2\sigma^4 V_0^2)$ in linear scale (left plot) and log scale (right plot), for two values of the disorder strength $V_0 m\sigma^2/\hbar^2 = 0.01$ (black curve) and 0.05 (red curve). The dashed green curve is $\pi(1 - x)/x^2$. Taken from [1].*

e) Discuss briefly what happens if a Gaussian laser beam (see section **Part 1.C**) is used in place of a square aperture?

D. Experimental results Using a cold atomic cloud of Rubidium and a speckle optical potential, Billy *et al.* [2] have observed 1D Anderson localization. Figure 3 shows the experimental results, where the measured localization length is plotted versus the disorder strength. The energy of the atoms was such that $k\sigma \approx 0.65$ and $\sigma \approx 0.26\mu$m. Comment these experimental results. What is the order of magnitude of the product $k\ell$? What will happen if the same experiment is performed in 2D or 3D?
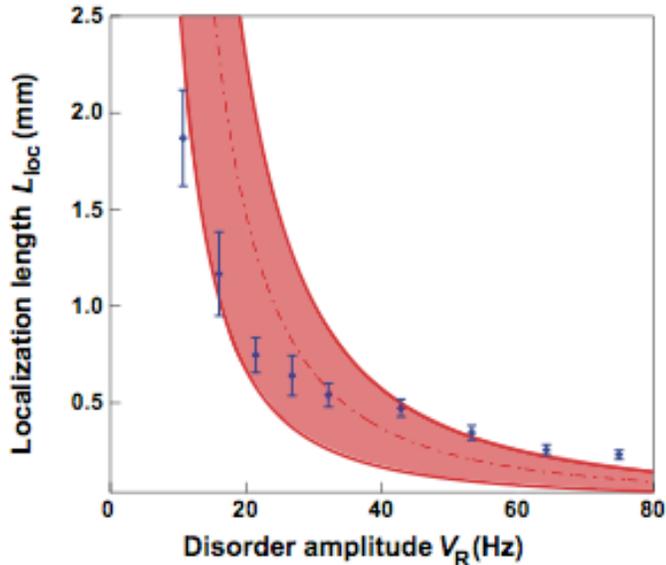

Figure 3: *Localization length for a cloud of cold Rubidium atoms in a disordered "speckle" optical potential, versus the strength V_0 (here noted V_R) of the disorder. Experimental parameters are such that $k\sigma \approx 0.65$ and $\sigma \approx 0.26\mu$m. The experimental results with error bars are the blue points, the prediction of eq.(13) shown by the red dash-dotted line. The shaded area reflects the uncertainties on the parameters. Taken from Ref. [2].*

References

[1] P. Lugan, A. Aspect, L. Sanchez-Palencia, D. Delande, B. Grémaud, C. A. Müller and C. Miniatura "*One-dimensional Anderson localization in certain correlated random potentials*", Phys. Rev. A **80**, 023605 (2009) [arxiv:0902.0107].

[2] J. Billy, V. Josse, Z. Zuo, A. Bernard, B. Hambrecht, P. Lugan, D. Clément, L. Sanchez-Palencia, P. Bouyer and A. Aspect, "*Direct observation of Anderson localization of matter-waves in a controlled disorder*", Nature **453**, 891 (2008).

<center>ANSWERS</center>

Part 1: speckle

A. Diffraction by a square aperture.– Because the plane wave is perpendicular to the aperture, the amplitude is uniform at $z = 0$, so that :

$$\mathcal{E}(k_x, k_y) = \mathcal{A}\, L^2\, \text{sinc}\left(\frac{k_x L}{2}\right)\, \text{sinc}\left(\frac{k_y L}{2}\right) \tag{14}$$

and

$$I(k_x, k_y) = \mathcal{A}^2\, L^4\, \text{sinc}^2\left(\frac{k_x L}{2}\right)\, \text{sinc}^2\left(\frac{k_y L}{2}\right) . \tag{15}$$

B. Diffraction by a diffusive plate.

1/ The phase $\phi(x,y)$ of the field is uniformly distributed, hence $\langle A(\vec{r})\rangle = \mathcal{A}\int_0^{2\pi}\frac{\mathrm{d}\phi}{2\pi}\,\mathrm{e}^{\mathrm{i}\phi} = 0$. Similarly for $\langle A(\vec{r})^2\rangle = 0$. The squared modulus is of course independent of the phase, so that only $\langle A(\vec{r})A^*(\vec{r})\rangle = |A(\vec{r})|^2 = \mathcal{A}^2$ survives.

Discretizing (1) as $\mathcal{E}(\vec{k}) = \eta^2\sum_{n=1}^N A(\vec{r}_n)\,\mathrm{e}^{\mathrm{i}\vec{k}\cdot\vec{r}_n}$ makes clear that the field is the sum of a large number of uncorrelated random variables. Hence we can apply the *central limit theorem*, which implies that the distribution of $\mathcal{E}$ is *Gaussian*. We have $\langle\mathcal{E}(\vec{k})\rangle = 0$, $\langle\mathcal{E}(\vec{k})^2\rangle = 0$ and

$$\langle|\mathcal{E}(\vec{k})|^2\rangle = \eta^4\sum_{n,m}\underbrace{\langle A(\vec{r}_n)A^*(\vec{r}_m)\rangle}_{\mathcal{A}^2\,\delta_{n,m}}\,\mathrm{e}^{\mathrm{i}\vec{k}\cdot(\vec{r}_n-\vec{r}_m)} = \eta^4 N\mathcal{A}^2 = \eta^2 L^2\mathcal{A}^2 = I_0$$

is the averaged intensity. If we decompose the electric field as $\mathcal{E} = \mathcal{X}+\mathrm{i}\,\mathcal{Y}$, we deduce from $\langle\mathcal{E}^2\rangle = \langle\mathcal{X}^2\rangle - \langle\mathcal{Y}^2\rangle + 2\mathrm{i}\,\langle\mathcal{X}\mathcal{Y}\rangle = 0$ that the two components are uncorrelated, $\langle\mathcal{X}\mathcal{Y}\rangle = 0$, with same variance $\langle\mathcal{X}^2\rangle = \langle\mathcal{Y}^2\rangle$ (isotropy). As a result the distribution of the electric field has the form $P_\mathcal{E}(\mathcal{E},\mathcal{E}^*) \propto \exp\left[-(1/2c)(\mathcal{X}^2+\mathcal{Y}^2)\right]$. Using that the averaged intensity is $\langle I\rangle = \langle|\mathcal{E}|^2\rangle = \langle\mathcal{X}^2\rangle + \langle\mathcal{Y}^2\rangle = 2\langle\mathcal{X}^2\rangle = 2c \overset{\text{def}}{=} I_0$ we finally deduce the form

$$\mathcal{P}_\mathcal{E}(\mathcal{E},\mathcal{E}^*) = \frac{1}{\pi I_0}\,\mathrm{e}^{-|\mathcal{E}|^2/I_0} . \tag{16}$$

2/ The distribution of the intensity is given by writing the field in polar coordinates $\mathcal{E} = \mathcal{R}\,\mathrm{e}^{\mathrm{i}\phi}$ and $P(I)\mathrm{d}I = \mathcal{R}\mathrm{d}\mathcal{R}\int_0^{2\pi}\mathrm{d}\phi\,\mathcal{P}_\mathcal{E}(\mathcal{E},\mathcal{E}^*) = \mathrm{d}(\mathcal{R}^2)/2\int\mathrm{d}\phi\,\mathcal{P}_\mathcal{E}(\mathcal{E},\mathcal{E}^*)$, hence the Rayleigh law

$$P(I) = \pi\,\mathcal{P}_\mathcal{E}(\mathcal{E},\mathcal{E}^*) = \frac{1}{I_0}\,\mathrm{e}^{-I/I_0} .$$

The most probable value is thus $I = 0$. The first moments are $\langle I\rangle = I_0$ and $\langle I^2\rangle = 2I_0^2$, hence $\text{Var}(I) = I_0^2 = \langle I\rangle^2$.

3/ *a*) Correlations are conveniently computed with the discrete formulation

$$\langle\mathcal{E}(\vec{k})\mathcal{E}^*(\vec{k}')\rangle = \eta^4\sum_{n,m}\langle A(\vec{r}_n)A^*(\vec{r}_m)\rangle\,\mathrm{e}^{\mathrm{i}\vec{k}\cdot\vec{r}_n-\mathrm{i}\vec{k}'\cdot\vec{r}_m} = \eta^4\mathcal{A}^2\sum_n\mathrm{e}^{\mathrm{i}(\vec{k}-\vec{k}')\cdot\vec{r}_n}$$

$$= \frac{I_0}{L^2}\int_{-L/2}^{L/2}\mathrm{d}x\int_{-L/2}^{L/2}\mathrm{d}y\,\mathrm{e}^{\mathrm{i}(\vec{k}-\vec{k}')\cdot\vec{r}}$$

<center>7</center>

so that :

$$\langle \mathcal{E}(\vec{k})\mathcal{E}^*(\vec{k}')\rangle = I_0 \operatorname{sinc}\left(\frac{\Delta k_x L}{2}\right)\operatorname{sinc}\left(\frac{\Delta k_y L}{2}\right) \tag{17}$$

b) $\mathcal{E}$ is Gaussian, therefore we can use Wick's theorem :

$$\langle I(\vec{k})\,I(\vec{k}')\rangle = \underbrace{\langle \mathcal{E}(\vec{k})\mathcal{E}^*(\vec{k})\rangle\langle \mathcal{E}(\vec{k}')\mathcal{E}^*(\vec{k}')\rangle}_{=I_0^2} + \underbrace{\langle \mathcal{E}(\vec{k})\mathcal{E}^*(\vec{k}')\rangle\langle \mathcal{E}(\vec{k}')\mathcal{E}^*(\vec{k})\rangle}_{=|\langle \mathcal{E}(\vec{k})\mathcal{E}^*(\vec{k}')\rangle|^2} + \underbrace{\langle \mathcal{E}(\vec{k})\mathcal{E}(\vec{k}')\rangle\langle \mathcal{E}^*(\vec{k})\mathcal{E}^*(\vec{k}')\rangle}_{=0}$$

(recall that there are $(2n-1)!!$ manners to make n pairs from $2n$ variables).

Higher moments : $\langle I^n\rangle = \langle \mathcal{E}\mathcal{E}^* \cdots \mathcal{E}\mathcal{E}^*\rangle$. There are $n!$ manners to pair the n amplitudes to the n complex amplitudes, thus $\langle I^n\rangle = n!\,\langle |\mathcal{E}|^2\rangle^n = n!\,I_0^n$, which are indeed the moments of the exponential law (2).

4/ *a*) To transform the angularly correlated intensity distribution into a spatially correlated one, the easiest way is to add a converging lens and look in the focal plane. A ray with direction $\vec{k}$ in the plane (xOz) characterized by the angle θ such that $\tan\theta = k_x/k_z \approx k_x/k_0$ is thus at position $x/f = \tan\theta$ in the focal plane, hence

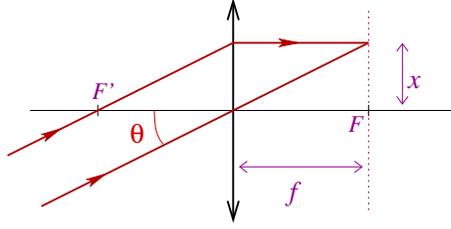$$\Delta\vec{r} \simeq \frac{\Delta\vec{k}}{k_0}\,f \tag{18}$$

Figure 4: *The atoms in the focal plane feel the disordered potential characterized by the correlations* (4).

b) The quasi-resonant laser field modifies the internal energy levels of the atom (light-shift). By energy conservation, this creates for the atom center of mass an effective potential proportional to the laser intensity $I(\vec{r})$ and inversely proportional to the detuning δ. The atom "feels" a potential $V(\vec{r}) \propto I(\vec{r})$, thus

$$\langle V(\vec{k})\,V(\vec{r}')\rangle \propto \langle I(\vec{k})\,I(\vec{k}')\rangle = I_0^2\left[1 + \operatorname{sinc}^2\left(\frac{\Delta x}{\sigma}\right)\operatorname{sinc}^2\left(\frac{\Delta y}{\sigma}\right)\right]$$

with $\sigma = 2f/(k_0 L) = f\lambda_0/(\pi L)$, where we made use of (18). Because of the paraxial approximation L/f must be small. The shortest achievable correlation length is of the order of λ_0.

C. Gaussian aperture.– A Gaussian beam arrives on the diffusive plate, $A(\vec{r}) = \mathcal{A}\exp[-\vec{r}^2/(2r_0^2)]$ (just before the plate). Using the Fraunhofer formula, we obtain that the correlation function of the field after the diffusive plate is

$$\langle \mathcal{E}(\vec{k})\mathcal{E}^*(\vec{k}')\rangle = \eta^2 \mathcal{A}^2 \int d\vec{r}\,e^{i(\vec{k}-\vec{k}')\cdot\vec{r}}\left(e^{-\vec{r}^2/(2r_0^2)}\right)^2 = \tilde{I}_0\,e^{-\Delta\vec{k}^2 r_0^2/4}$$

with $\tilde{I}_0 = \pi r_0^2 \eta^2 \mathcal{A}^2$. The intensity correlations are

$$\langle I(\vec{k})\, I(\vec{k}') \rangle = \tilde{I}_0^2 \left[1 + \mathrm{e}^{-\Delta \vec{k}^2 r_0^2/2} \right] \ .$$

The potential correlations for the atoms are Gaussian as well.

Part 2 : localization of an atom

A. Structure of the Green functions

1/ *a)* Free Green function : $\widetilde{G}_0(k, k'; E) = \langle \phi_k | (E - H_0 + \mathrm{i}0^+)^{-1} | \phi_{k'} \rangle = \delta_{k,k'} (E - \varepsilon_k + \mathrm{i}0^+)^{-1}$ with $\varepsilon_k = \hbar^2 k^2/(2m)$.

b) The corresponding expression in x-space is

$$G_0(x, x'; E) = \int \frac{\mathrm{d}k}{(2\pi)} \frac{\mathrm{e}^{\mathrm{i}k(x-x')}}{E - \varepsilon_k + \mathrm{i}0^+} = \frac{m}{\mathrm{i}\,\hbar^2 k_E}\, \mathrm{e}^{\mathrm{i}k_E |x-x'|} \qquad \text{where } k_E \stackrel{\mathrm{def}}{=} \sqrt{2mE}/\hbar. \quad (19)$$

Integral is easily computed with residue's theorem by closing the contour by a semi-circle of radius $\mathcal{R} \to \infty$ in the upper (lower) plane for $(x - x') > 0$ (< 0, resp.).

2/ The Green function is defined by $G(E) = 1/(E - H + \mathrm{i}0^+)$ and satisfies the Dyson equation $G = G_0 + G_0 V G$.

   The Green function is different for every realization of the disorder. Its average describes the spatio-temporal evolution of the average field (NOT the average intensity). The Dyson equation for the averaged Green function is

$$\overline{G} = G_0 + G_0\, \Sigma\, \overline{G}\,, \tag{20}$$

and corresponds to a reorganisation of the perturbative expansion in terms of *irreducible* diagrams. We deduce Eq. (8).

   Translational invariance is restored after disorder averaging, which implies that $\overline{G}$ and Σ are diagonal in the k-representation : $\langle \phi_k | \Sigma(E) | \phi_{k'} \rangle = \Sigma(k; E)\, \delta_{k,k'}$.

3/ In real space we have

$$\overline{G}(x, x'; E) = \int \frac{\mathrm{d}k}{(2\pi)} \frac{\mathrm{e}^{\mathrm{i}k(x-x')}}{E - \varepsilon_k - \Sigma(k; E)} \ . \tag{21}$$

In the weak disorder limit, $|\Sigma(k; E)| \ll |E|$, the poles $k_{\pm}$ of the integrand, i.e. zeros of $E - \varepsilon_k - \Sigma(k; E)$, are only slightly shifted and remain close to $\pm k_E$. Thus, assuming that Σ is a smooth function we can simply perform the substitution $\Sigma(k; E) \to \Sigma(k_E; E)$. Expanding the position of the poles for small Σ gives

$$k_{\pm} \simeq \pm \underbrace{\sqrt{\frac{2m}{\hbar^2}(E - \mathrm{Re}\,\Sigma)}}_{\stackrel{\mathrm{def}}{=} \tilde{k}_E \simeq k_E} \left(1 - \frac{\mathrm{i}\,\mathrm{Im}\,\Sigma}{2E} \right)$$

we see that the calculation of $\overline{G}$ is the same as the one of G_0, provided the substitution

$$k_E \longrightarrow \tilde{k}_E - \frac{\mathrm{i}m}{\hbar^2 k_E}\, \mathrm{Im}\left[\Sigma(k_E; E) \right] \ .$$

Making this substitution in Eq. (19), we obtain the structure

$$\overline{G}(x, x'; E) \approx G_0(x, x'; E) \exp\left(-\frac{|x - x'|}{2\ell(E)}\right)$$

with

$$\frac{1}{\ell(E)} \overset{\text{def}}{=} -\frac{2m}{\hbar^2 k_E} \operatorname{Im}\left[\Sigma(k_E; E)\right]$$

$\operatorname{Re}\Sigma$ appears as a (negative) correction to the energy due to the average disorder. It is a simple shift of the energy axis. $\ell(E)$ is of course the mean free path for atom with energy E. It corresponds to the typical distance between two collisions on the disorder ($\ell(E)/v_E$ is the lifetime of the plane wave).

B. Born approximation.

1/ Cf. lecture's notes. The perturbative expansion of $\overline{G}$ reads

$$\overline{G} = G_0 + G_0 \overline{V} G_0 + G_0 \overline{V G_0 V} G_0 + \cdots$$

2/ We deduce $\Sigma = V_0 + \cdots$ at lowest order. This is a (trivial) shift in energy due to $\langle V \rangle = V_0 \neq 0$.

3/ Next leading order gives $\Sigma = V_0 + \overline{V G_0 V} + \cdots$. We sandwich the expression between two plane waves :

$$\Sigma(k; E) = V_0 + \langle \phi_k | \overline{V G_0(E) V} | \phi_k \rangle + \cdots = V_0 + \sum_{k'} \overline{|\langle \phi_k | V | \phi_{k'} \rangle|^2} \, G_0(k'; E) + \cdots$$

Introducing the correlation function of the disorder we easily get

$$\Sigma(k; E) \simeq V_0 + V_0^2 \int \frac{\mathrm{d}k'}{2\pi} G_0(k'; E) \, \mathcal{C}(k - k')$$

The important part is the imaginary part. Using that $\operatorname{Im}[G_0(k'; E)] = -\pi \, \delta(E - \varepsilon_{k'})$ we finally obtain

$$\operatorname{Im}\left[\Sigma(k; E)\right] \simeq -\frac{mV_0^2}{2\hbar^2 k_E} [\mathcal{C}(k - k_E) + \mathcal{C}(k + k_E)] \,.$$

As we have seen the elastic mean free path involves the "on-shell" self energy

$$\operatorname{Im}\left[\Sigma(k_E; E)\right] \simeq -\frac{mV_0^2}{2\hbar^2 k_E} [\mathcal{C}(0) + \mathcal{C}(2k_E)] \,,$$

where the two terms are interpreted as the contributions of *forward* scattering $\propto \mathcal{C}(0)$ and *backward* scattering $\propto \mathcal{C}(2k_E)$.

This leads to the formula for the *elastic mean free path*

$$\frac{1}{\ell(E)} \simeq \frac{m^2 V_0^2}{\hbar^4 k_E^2} [\mathcal{C}(0) + \mathcal{C}(2k_E)] \,. \tag{22}$$

C. Localization length.

1/ A Fourier transform.– The Fourier transform of $\mathrm{sinc}(xa/2)$ is the "door" $\int \mathrm{d}x \, \mathrm{sinc}(xa/2) \, \mathrm{e}^{-\mathrm{i}kx} = (2\pi/a)\Pi_a(k)$ where $\Pi_a(k) = \theta_{\mathrm{H}}(a/2 - |k|)$ (the converse is easy to check). The Fourier transform of $\mathrm{sinc}^2(xa/2)$ is therefore the convolution $(2\pi/a^2)(\Pi_a * \Pi_a)(k)$. In order to avoid an explicit calculation, we now make two remarks :
(*i*) $(\Pi_a * \Pi_a)(0) = \int \mathrm{d}k \, \Pi_a(k) = a$.
(*ii*) the overlap between $\Pi_a(k')$ and $\Pi_a(k - k')$ increases linearly with k and vanishes for $|k| > a$, hence $(\Pi_a * \Pi_a)(k) = a(1 - |k|/a) \, \theta_{\mathrm{H}}(1 - |k|/a)$.

We deduce

$$\mathcal{C}(k) = \pi \, \sigma \left(1 - \frac{|k|\sigma}{2}\right) \theta_{\mathrm{H}}\left(1 - \frac{|k|\sigma}{2}\right) .$$

2/ *a*) In 1D, any disorder with short range correlations leads to strong localization of all eigenstates (no mobility edge). In the lecture we have noticed that in 1D the mean free path and the localisation length are related by $\xi_{\mathrm{loc}} \simeq 2\ell$ (in the weak disorder regime), which is valid for $\mathcal{C}(k) = $ cste, when scattering is isotropic (forward=backward scattering), i.e. $\langle V(x)V(x') \rangle \propto \delta(x - x')$. Hence, the perturbative calculation of the Green function has provided a (perturbative) formula for the localization length.

b) When scattering is not isotropic (i.e. forward and backward scattering have different probabilities), we have to take into account that *only backward scattering leads to localization*. This remark explains that one should perform the subtitution $\mathcal{C}(0) \to \mathcal{C}(2k)$ in the formula for the elastic mean free path :

$$\frac{1}{\xi_{\mathrm{loc}}} \simeq \frac{1}{2\ell_{\mathrm{tr}}} \simeq \frac{m^2 V_0^2}{\hbar^4 k_E^2} \mathcal{C}(2k_E)$$

where we have used (22).

c) Using the expression of the correlation function computed previously, $\mathcal{C}(2k) = \pi \, \sigma \, (1 - |k|\sigma) \, \theta_{\mathrm{H}}(1 - |k|\sigma)$, we end with

$$\boxed{\frac{1}{\xi_{\mathrm{loc}}} \simeq \frac{\pi \, m^2 \, V_0^2 \, \sigma}{\hbar^4 k_E^2} \, (1 - k_E \sigma) \, \theta_{\mathrm{H}}(1 - k_E \sigma)}$$

d) The figure shows that the theoretical prediction is not too bad. As expected for an expansion in powers of the disorder strength, it works better at small V_0. It always fail at low-k (small energy) because the self-energy $\Sigma(E)$ is no longer much smaller than E.

The *perturbative* expression of the inverse localization length vanishes when the energy is above a threshold, i.e. for $k_E > 1/\sigma$. This should however not be interpreted as a mobility edge! In 1D, localization takes place **at any energy**. A calculation involving higher order contributions to Σ indeed predicts a finite localization length [1], in agreement with the numerics. For small V_0, the vanishing of the second order contribution manifests as the "accident" (step like behaviour). As expected, the step is less pronounced for larger V_0.

D. Experimental results.— The observed localization length decays with increasing V_0, but the $1/V_0^2$ prediction is not very good. This is partly because one should include higher-order contributions in V_0 and partly due to experimental problems (difficulty of measuring long localization lengths at small V_0, residual atom-atom interaction killing

localization at large V_0). As ξ_{loc}/σ is at least 100, the formula shows that $k\ell$ is at least as large. In 2D, this would result in a huge localization length, scaling like $\exp(k\ell)$, thus larger than the size of the universe ; in 3D, it would be on the diffusive side of the Anderson metal-insulator transition.