Introduction to Many-Body Localization

Physics in the middle of the spectrum

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Outline

- Introduction
- A toy model for MBL
- Toy model of the toy model
- Experiments
- (If time allows) More features
- Open issues & Conclusions

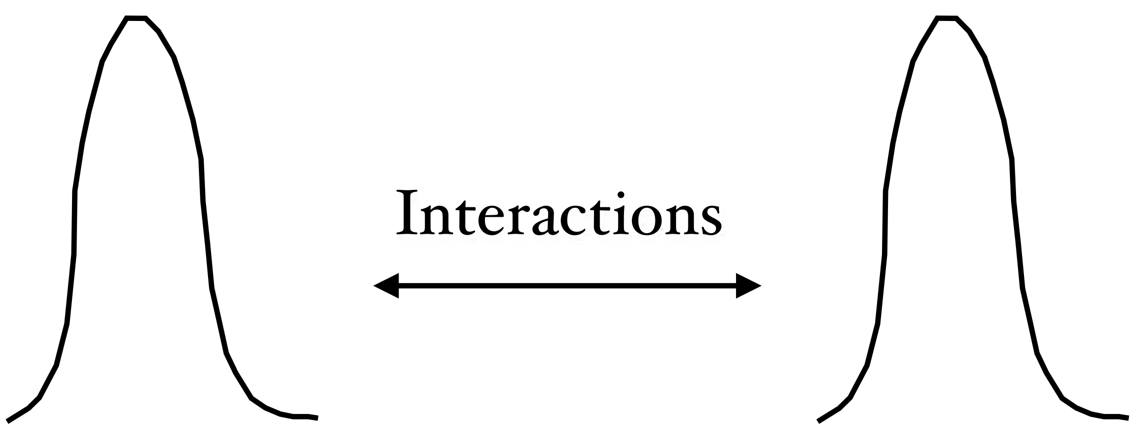
What is Many-Body Localization ?

«Many-body localization» (MBL)

A new distinct dynamical phase of matter, which does not self-thermalize

- Characteristics:
  - Zero DC conductivity at finite temperature
  - Low entanglement
  - Anderson localization + dephasing

- Key ingredients : Disorder + Interactions, Isolated system



In general, expect interactions to induce transport and to thermalize an isolated localized system

Why Many-Body Localization?

- Why is it an interesting problem? Mostly fundamental questions:
  - Does a closed quantum system self-thermalize?
  - Is there a perfect insulator at finite temperature?
  - What happens to Anderson localization in presence of interactions?
  - …

- Why is it a difficult problem?
  - All the tough ingredients are there : Quantum Many-Body interactions, disorder, out-of-equilibrium
  - Absence of thermalization: can't use thermodynamic ensembles!
  - Usual condensed matter methods geared towards low-energy properties
  - Too many papers !!

PREQUEL

Eigenstate Thermalization Hypothesis

Thermalization of isolated systems

- Quench protocol: Evolve initial state with a (many-body) Hamiltonian $|\Psi(t)\rangle = \exp(-iHt)|\Psi_0\rangle$

Q. : Does the system reach thermal equilibrium ?

- Expand $|\Psi_0\rangle = \sum_i a_n |n\rangle$ in eigenbasis of $H = \sum_n E_n |n\rangle\langle n|$
- Time-evolved observable (generic Hamiltonian)

$$\langle \mathcal{O}(t) \rangle = \sum_{n,n'} a^*_{n'} a_n e^{-i(E_{n'}-E_n)t} \mathcal{O}_{nn'} \xrightarrow{t\to\infty} \sum_n |a_n|^2 \mathcal{O}_{nn}$$

'Diagonal ensemble'

- Eigenstate thermalization hypothesis (ETH)

Deutsch, Srednicki, Rigol

$\langle n|\mathcal{O}|n\rangle \simeq \langle n'|\mathcal{O}|n'\rangle = \mathcal{O}(E)$ $\quad |n\rangle, |n'\rangle$ in the same energy shell

$\langle n|\mathcal{O}|n'\rangle$ vanish in the thermodynamic limit and for few-body observables

- ETH implies thermalization

$$\langle \mathcal{O}(t\to\infty) \rangle = \mathcal{O}(E) = \mathcal{O}(T)$$

$$E = \langle \Psi_0 | H | \Psi_0 \rangle$$

$$E = \langle H \rangle_T$$

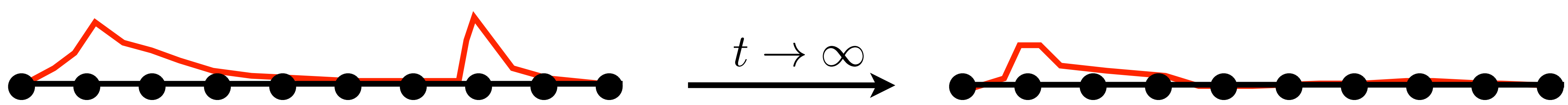
Consequences & Exceptions

- Each eigenstate is thermal, «knows» equilibrium

$$\rho(0) = |n\rangle\langle n| = \rho(t) = \rho^{eq}(T_n) \qquad E_n = \langle H \rangle_{T_n}$$

- Memory of initial conditions is lost
- ETH is a «justification» of the microcanonical ensemble at the individual eigenstate level
- Exceptions to ETH : Integrable systems

  Anderson (single-particle) localization



Many-Body Localized (MBL) systems

Old problem — revived by an enormous amount of contributions!

Anderson, Fleishmann, Shepelyansky...

'Old' Reviews (2015): Nandkishore & Huse, Altman & Vosk

New reviews upcoming (Annalen der Physik)

SEASON 1

Let's start with a toy model ...

Toy model to understand MBL

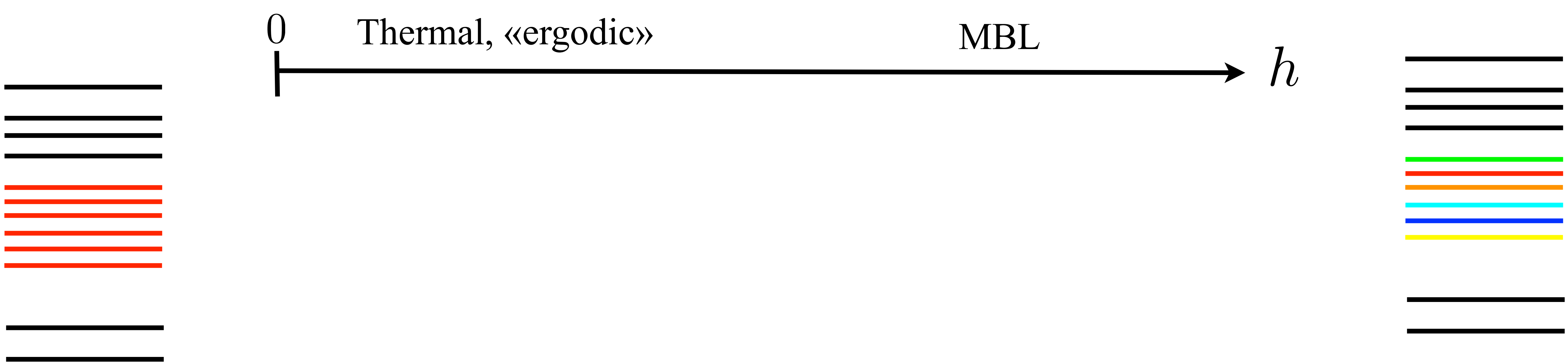
- XXZ Spin 1/2 chain in a random magnetic field

$$H = \sum_i S_i^x S_{i+1}^x + S_i^y S_{i+1}^y + \Delta S_i^z S_{i+1}^z - \sum_i h_i S_i^z$$

$\Delta \neq 0$

$h_i \in [-h, h]$

- Infinite disorder : eigenstates are fully localized product-states, no entanglement
- $\Delta = 0$: non-interacting case. Maps to 1d Anderson model
- Branch small interaction : perturbative calculations Gornyi *et al.*, Basko *et al.* indicate that thermalization does not occur: states keep localized, no spin or energy transport
- Beyond perturbation (numerics): localization can survive interactions



Eigenstates look all the same (~ Random Matrix Theory)

Eigenstates all different

For a given energy density (say middle of spectrum)

ETH states versus MBL states

Ergodic states

Follow ETH

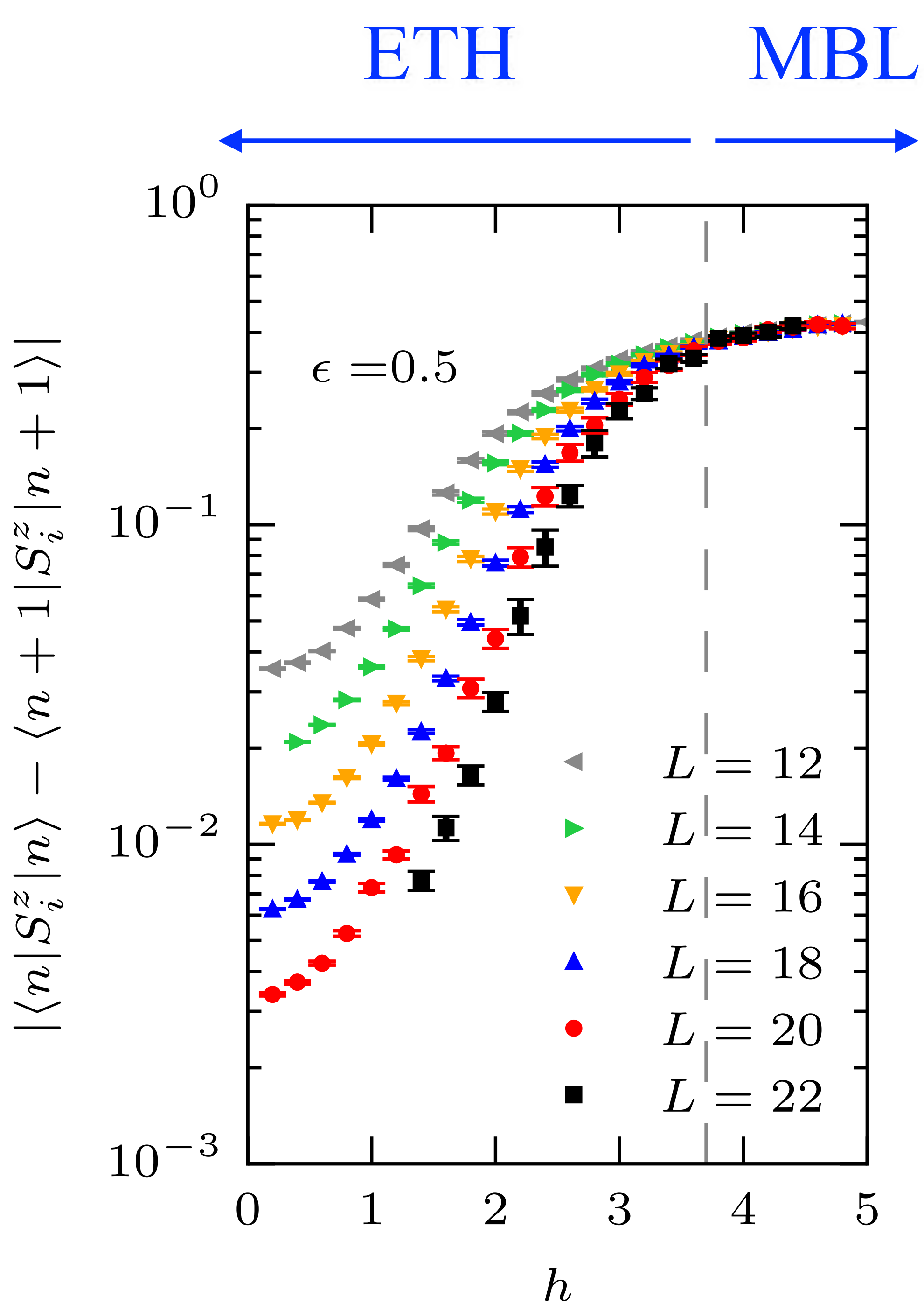
Observables are the same within the same energy shell

MBL states

Violate stat. mech.

Observables differ from eigenstate to eigenstate

Difference of local magnetization between consecutive eigenstates



ETH states versus MBL states

Ergodic states

Follow ETH

Observables are the same within the same energy shell

Random matrix statistics

MBL states

Violate stat. mech.

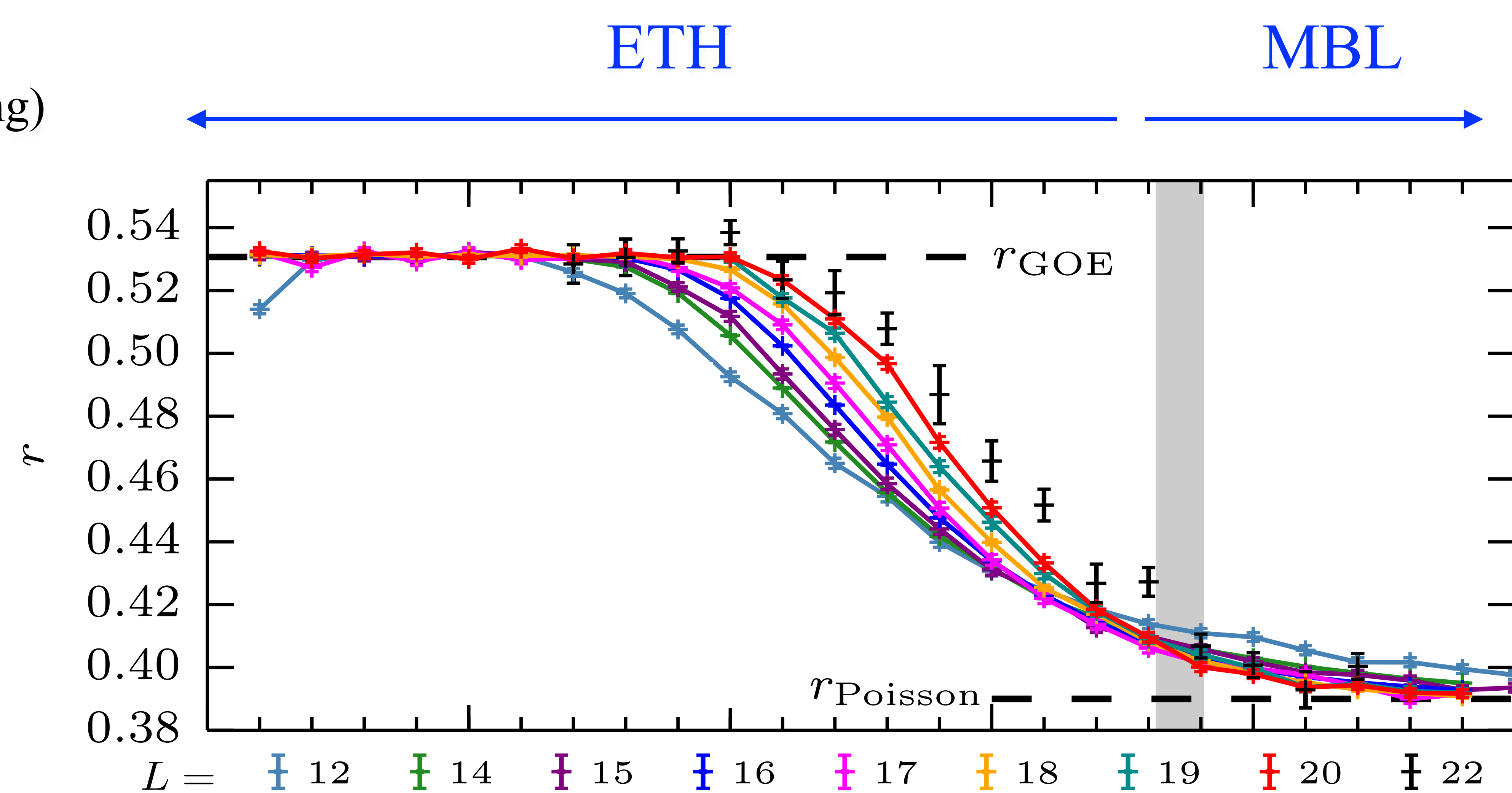
Observables differ from eigenstate to eigenstate

Integrable (Poisson) statistics

Gap ratio (avoids unfolding)

$$g_n = |E_n - E_{n-1}|$$

$$r = \frac{\min(g_n, g_{n+1})}{\max(g_n, g_{n+1})}$$



ETH states versus MBL states

Ergodic states	MBL states
Follow ETH	Violate stat. mech.
Observables are the same within the same energy shell	Observables differ from eigenstate to eigenstate
Random matrix statistics	Integrable (Poisson) statistics
Eigenstates occupy all configuration space	No delocalization

Localization of a wave-function in a basis $|n\rangle = \sum_i n_i |i\rangle$ $\quad p_i = |\langle n|i\rangle|^2$ $\quad \{|i\rangle\} = \{S^z\}$ basis

Participation entropies $S_1^p = -\sum_i p_i \ln(p_i)$ $\quad S_q^p = \frac{1}{1-q} \ln \sum_i p_i^q$ = ln (IPR)

ETH states versus MBL states

Ergodic states

Follow ETH

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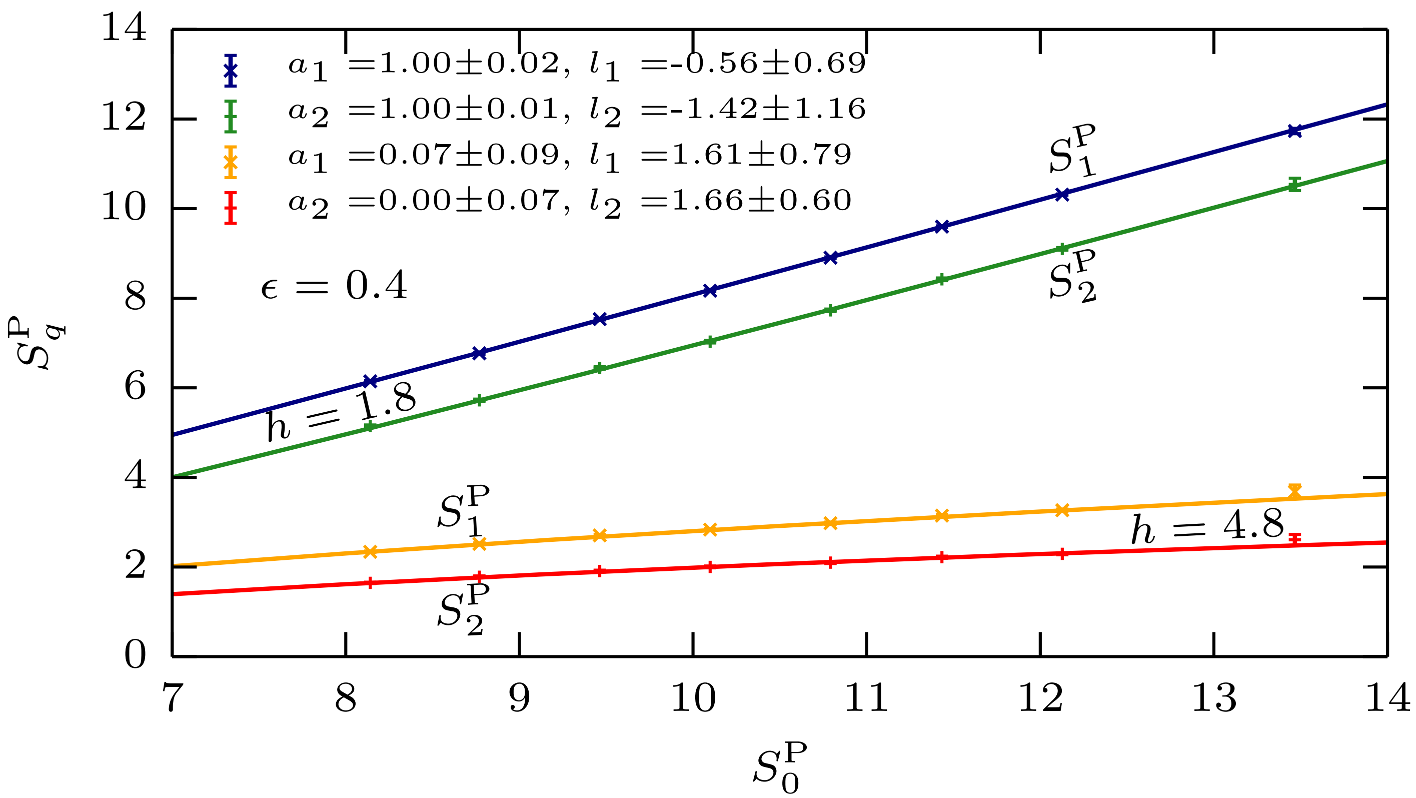
$$S_1^p = -\sum_i p_i \ln(p_i)$$

$$S_q^p = \frac{1}{1-q} \ln \sum_i p_i^q$$ = ln (IPR)



ETH: $S_q^p = a_q S_0^p - \ldots$ $a_q \simeq 1$

Eigenstates are completely delocalized

MBL: $a_q \ll 1$ or $S_q^p = l_q \ln(S_o^p) + \ldots$

Eigenstates are 'marginally' delocalized

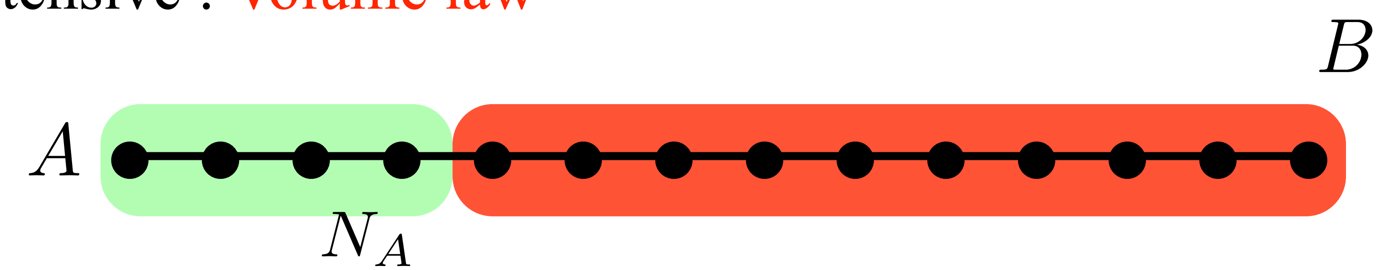
ETH states versus MBL states

Ergodic states	MBL states
Follow ETH	Violate stat. mech.
Observables are the same within the same energy shell	Observables differ from eigenstate to eigenstate
Random matrix statistics	Integrable (Poisson) statistics
Eigenstates occupy all configuration space	No delocalization
Volume Law for entanglement	Area law for entanglement

ETH : Entanglement entropy of eigenstates is extensive : Volume law

$$\rho_A = \mathrm{Tr}_B \rho = \rho_A^{\mathrm{eq}}(T_n)$$

$$S_A = -\mathrm{Tr}_A \rho_A \log \rho_A \propto N_A \quad \text{if} \quad T_n \neq 0$$



System is its own bath: B acts a thermal bath for A

MBL : states have low entanglement $S_A/N_A \to 0$: Area law

NB1 : Same entanglement structure as ground-states

NB2: MBL states efficiently represented as matrix-product states

Abanin, Vidal *et al.*

Eisert *et al.*

Pekker, Clark

ETH states versus MBL states

Ergodic states

Follow ETH

Observables are the same within the same energy shell

Random matrix statistics

Eigenstates occupy all configuration space

Volume Law for entanglement

MBL states

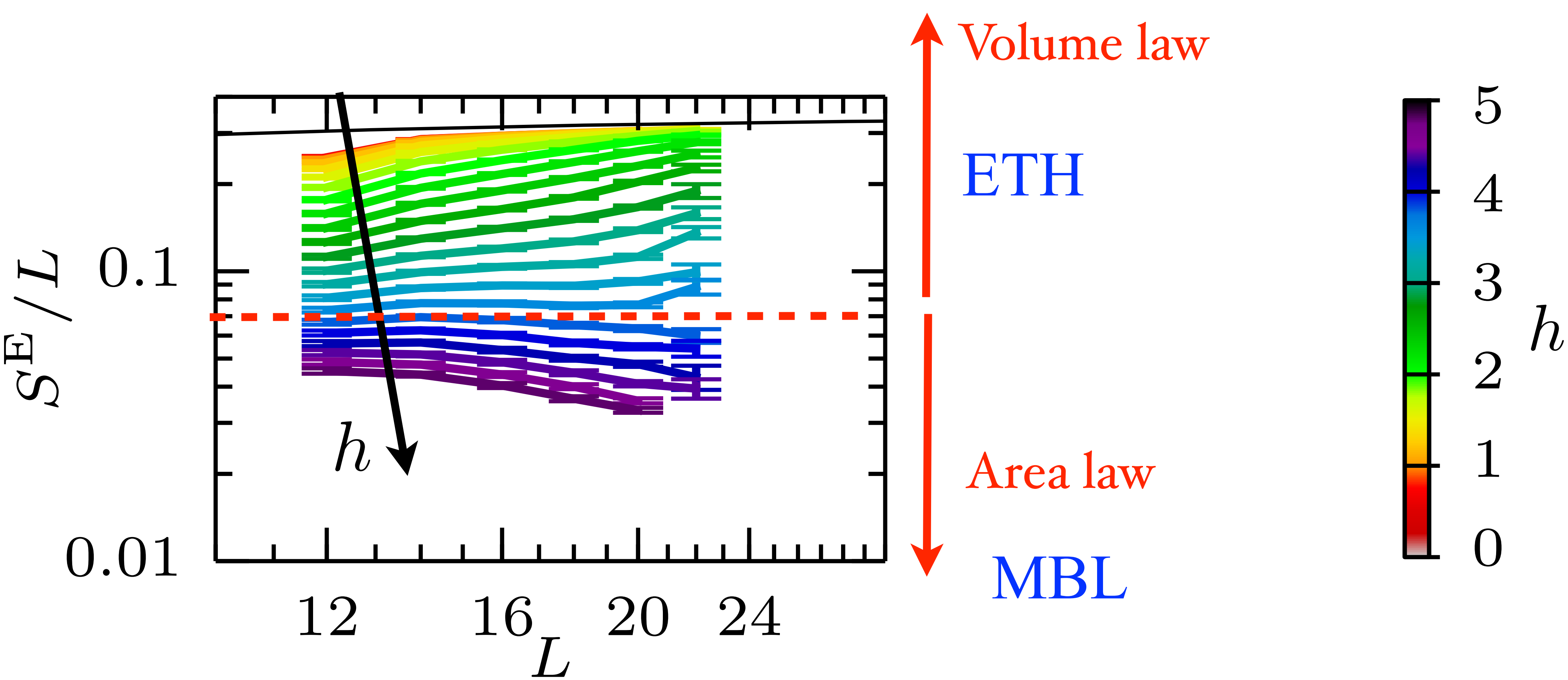
Violate stat. mech.

Observables differ from eigenstate to eigenstate

Integrable (Poisson) statistics

No delocalization

Area law for entanglement



ETH states versus MBL states

Ergodic states	MBL states
Follow ETH	Violate stat. mech.
Observables are the same within the same energy shell	Observables differ from eigenstate to eigenstate
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Volume Law for entanglement	Area law for entanglement

In a nutshell: MBL states do not follow ETH

MBL phase is not thermal

Crucial to work in the « eigenstate ensemble »

Each eigenstate has its own life

Dynamics

Ergodic phase

'Metal'

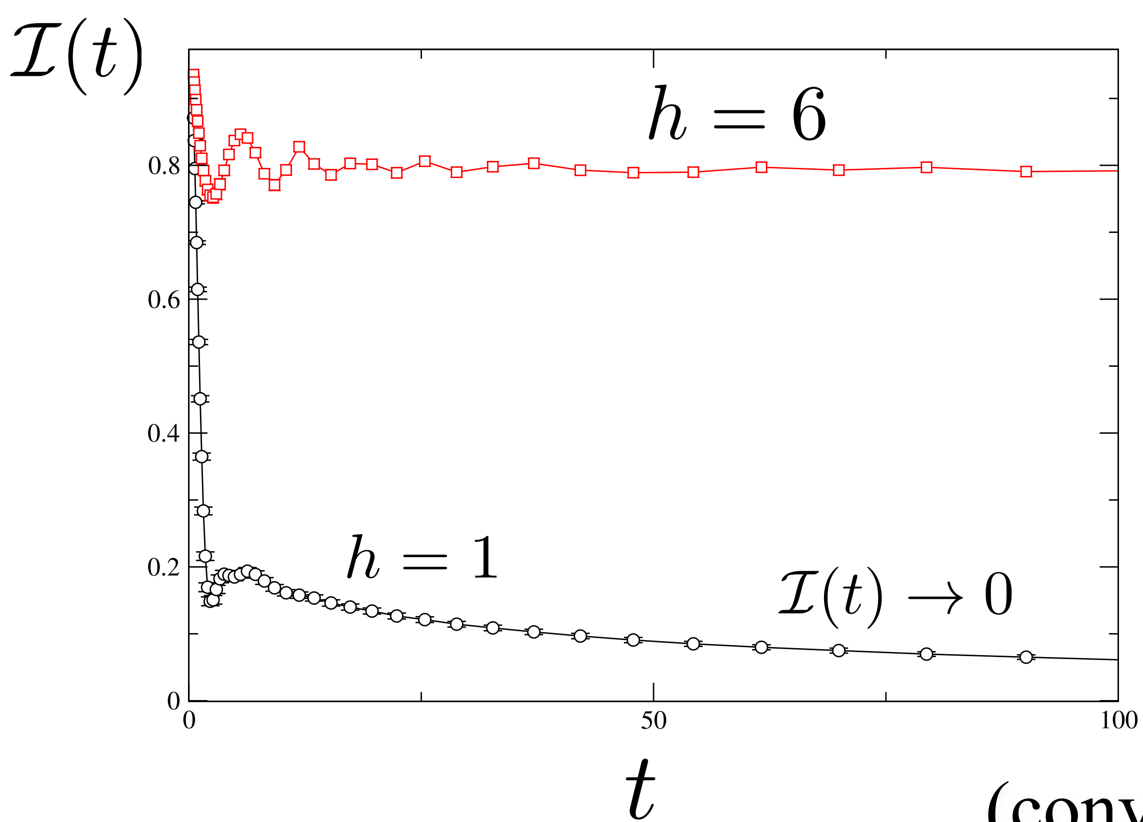
No memory of initial state

MBL phase

Not Anderson insulator either!

Memory, Revivals

Quench procedure : start from an initial CDW state (Néel), follow the imbalance (staggered magnetization)



$\mathcal{I}(t \to \infty) \neq 0$

MBL: memory of the initial state even at infinite time

Metal: no memory (converges to thermal ensemble)

Dynamics

Ergodic phase

'Metal'

No memory of initial state

Transport

MBL phase

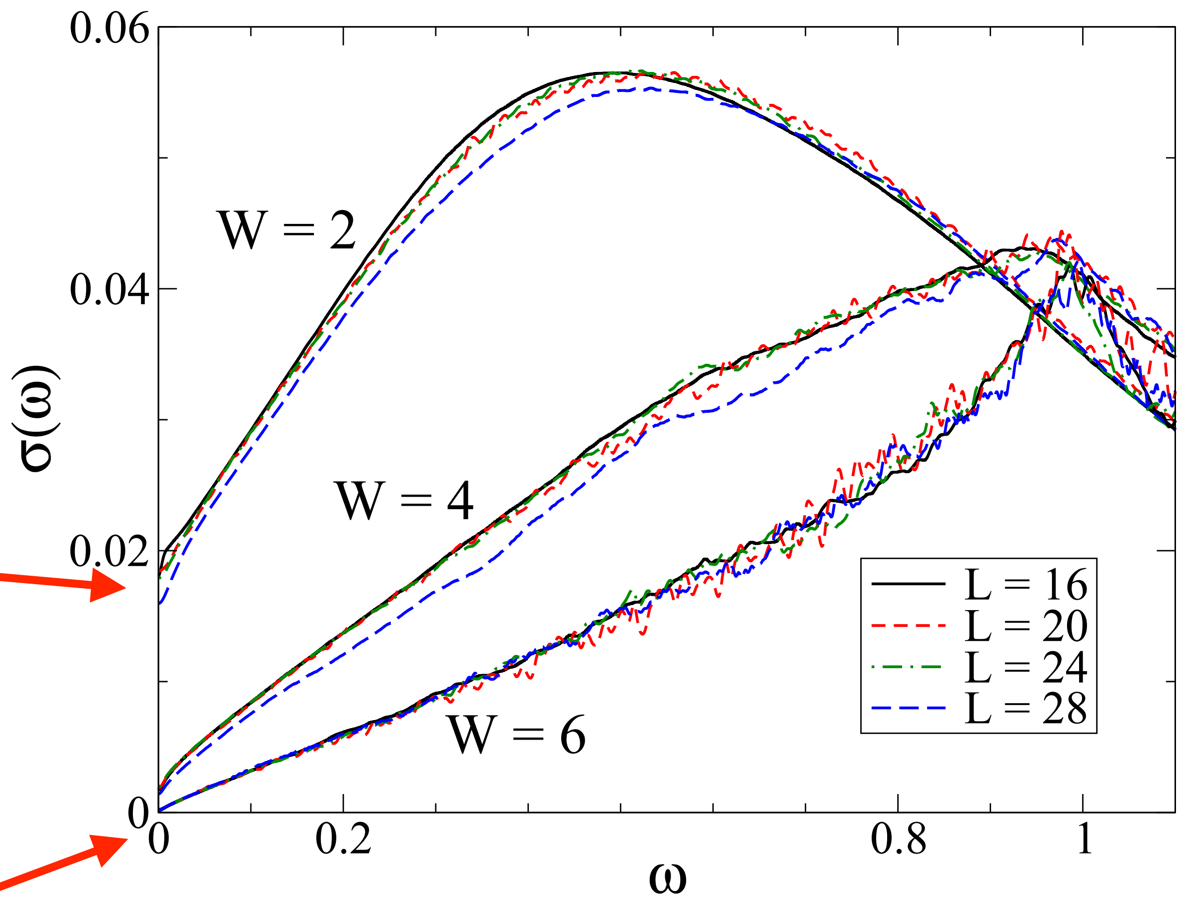
Not Anderson insulator either!

Memory, Revivals

No transport

Barisic *et al.*

Metal : Finite DC conductivity



MBL : Zero DC conductivity, even at infinite temperature

$$\sigma(\omega) = T\sigma(\omega) = 1/L \int_0^\infty dt \exp(i\omega t) \mathrm{Re}\langle j(t)j(0)\rangle$$

$$j = \frac{i}{2}\sum_i (S_i^+ S_{i+1}^- - S_i^- S_{i+1}^+)$$

Dynamics

Ergodic phase

'Metal'

No memory of initial state

Transport

Entanglement spreads fast

MBL phase

Not Anderson insulator either!

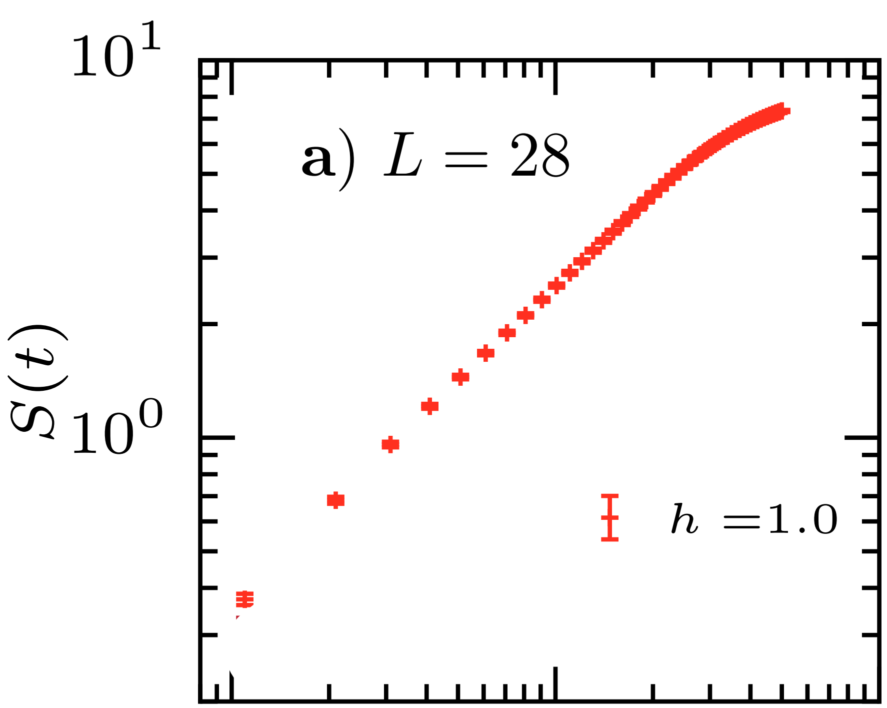
Memory, Revivals

No transport

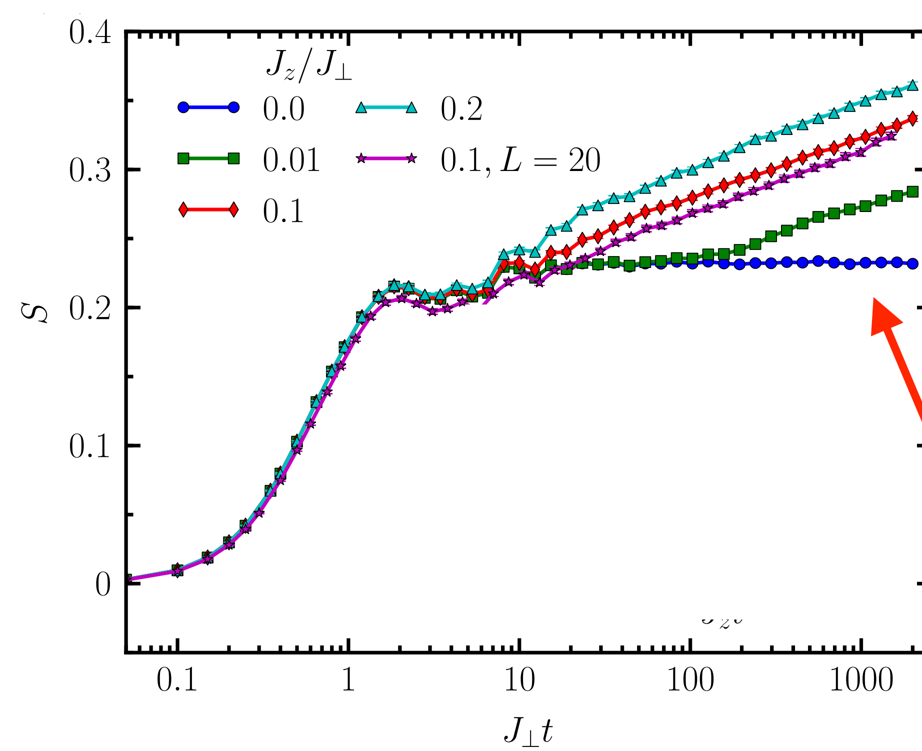
Entanglement spreads, but slowly

Quench from an initial product state : follow entanglement growth $S(t) = -\mathrm{Tr}\rho(t) \ln \rho(t)$

ETH : Ballistic growth $S(t) \propto t$



MBL: Logarithmic spread $S(t) \propto \log(t)$



Znidaric *et al.*

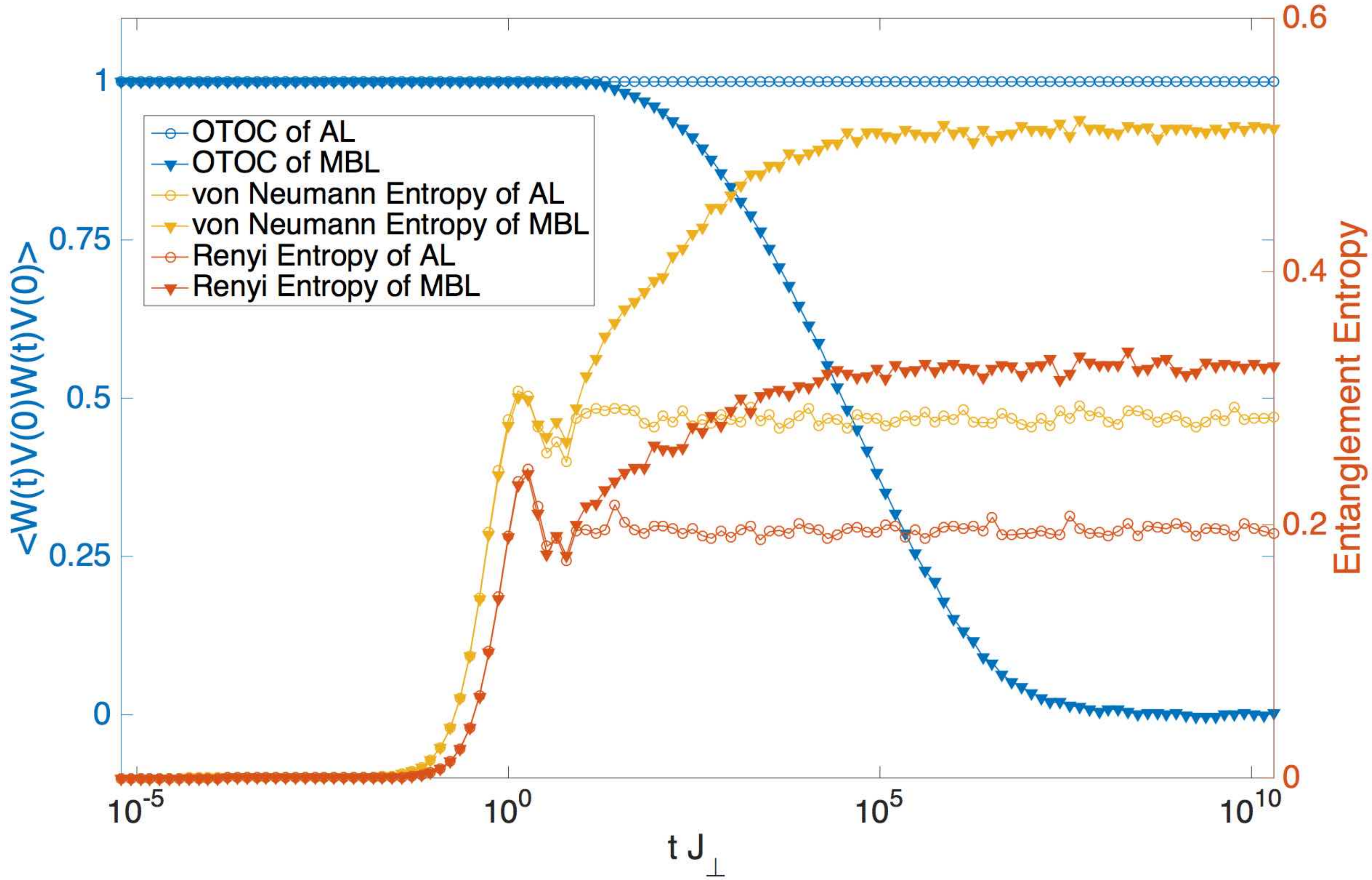
Bardarson *et al.*

Anderson : $S(t)$ bounded

Out-of-time ordered correlators

$\langle A^\dagger(t) B^\dagger(0) A(t) B(0) \rangle$

- OTOC quantifies 'scrambling' of quantum information (~ probability of attempting to recover information via local operations)



Fan *et al.*

PREVIOUSLY

From Nandkishore & Huse

ETH	Anderson	MBL
Thermal phase	Single-particle localized	Many-body localized
Memory of initial conditions 'hidden' in global operators at long times	Some memory of local initial conditions preserved in local observables at long times	Some memory of local initial conditions preserved in local observables at long times.
ETH true	ETH false	ETH false
May have non-zero DC conductivity	Zero DC conductivity	Zero DC conductivity
Continuous local spectrum	Discrete local spectrum	Discrete local spectrum
Eigenstates with volume-law entanglement	Eigenstates with area-law entanglement	Eigenstates with area-law entanglement
Power-law spreading of entanglement from non-entangled initial condition	No spreading of entanglement	Logarithmic spreading of entanglement from non-entangled initial condition
Dephasing and dissipation	No dephasing, no dissipation	Dephasing but no dissipation

SEASON 2

Toy model of the toy model

Phenomenology : ‘Fixed point’ of MBL

- Exact results Imbrie , phenomenology Huse *et al.*, Abanin *et al.* , perturbative results, Ros *et al.* strong disorder RG: Altman & Vosk

Quasi-local unitary transform can «diagonalize» the Hamiltonian

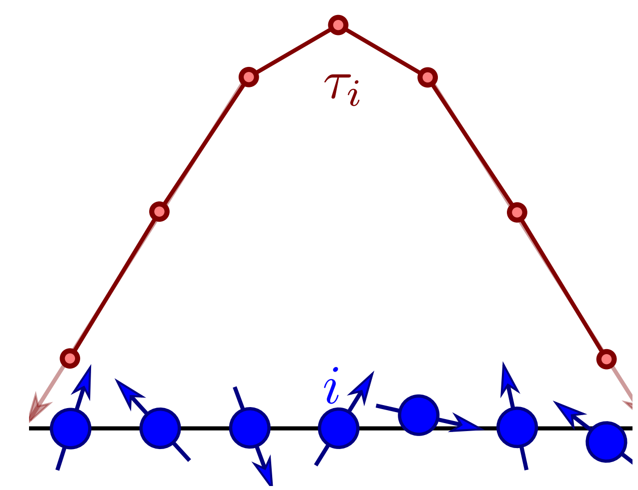
$$H_{\mathrm{l-bits}} \equiv H = -\sum_i h_i \tau_i^z + \sum_{i,j} J_{ij} \tau_i^z \tau_j^z + \sum_{i,j,k} J_{ijk} \tau_i^z \tau_j^z \tau_k^z + \dots$$

Exponential decay of coupling $J_{i_1,\dots,i_k} \sim \exp(-\frac{\max r_{i_1,i_2} \dots r_{i_j,i_k}}{\xi})$

- Localized bits (l-bits) $\tau_i^z \simeq \hat{Z}_i(h)\sigma_i^z + \text{exponential tail}$

$U\tau_i^z U^\dagger$ form a complete set of local integral of motions

MBL ~ integrable system



- Existence of l-bits now often seen as the definition of MBL
- Almost all features of MBL can be tested / understood in the l-bits language
- Various way of constructing l-bits: perturbation, time-averaged observables, real-space RG …

Imbrie, Ros, Scardicchio

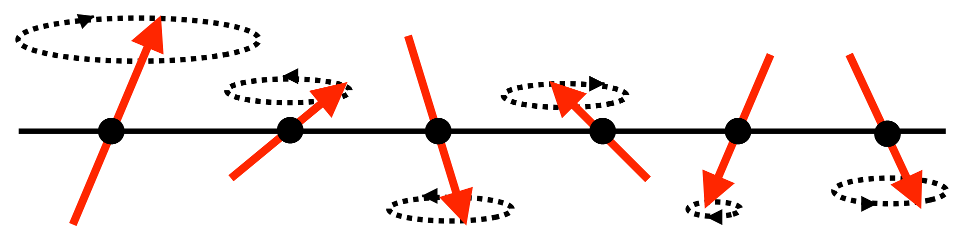
Dynamics & Difference with Anderson localization

- Anderson localized and MBL phases share similar features : no transport, integrable (Poisson) statistics, area law entanglement …

- Difference is dephasing: can be traced back to the exponential decaying interactions and tails

$$H_{\text{l-bits}} \equiv H = -\sum_i h_i \tau_i^z + \sum_{i,j} J_{ij} \tau_i^z \tau_j^z + \sum_{i,j,k} J_{ijk} \tau_i^z \tau_j^z \tau_k^z + \ldots$$

- Dynamics of l-bits in MBL phase: precession around z axis (τ^z conserved) with a rate due to interactions with other l-bits.



- For generic initial unentangled states : off-diagonal elements of reduced density matrix decay as a power law with time : dephasing

Serbyn *et al.*

$$\langle \tau^z(t) \rangle = \langle \tau^z(0) \rangle \qquad \langle \tau^x(t) \rangle \sim t^{-a} \qquad a \sim \xi$$

- No « spin-flip » term τ^x in the l-bits Hamiltonian: « no dissipation »

- Explains differences between Anderson & MBL : log growth of entanglement, more complex Out-of-Time-Order Correlations …

- Helps designing spin-echo protocols to detect / manipulate MBL

Serbyn *et al.*

Experiments

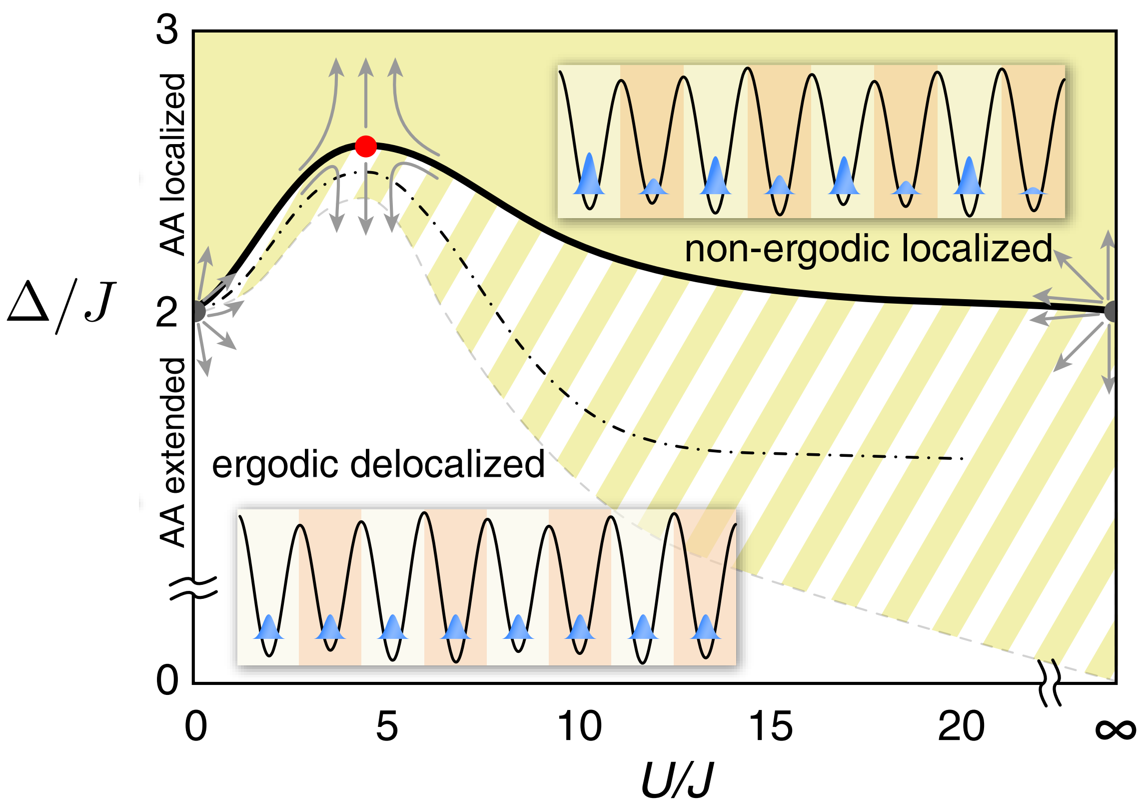
Experiments : Fermions in 1d

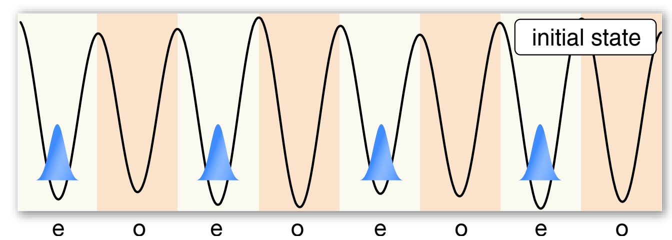
Schreiber *et al.*, Science (2015)

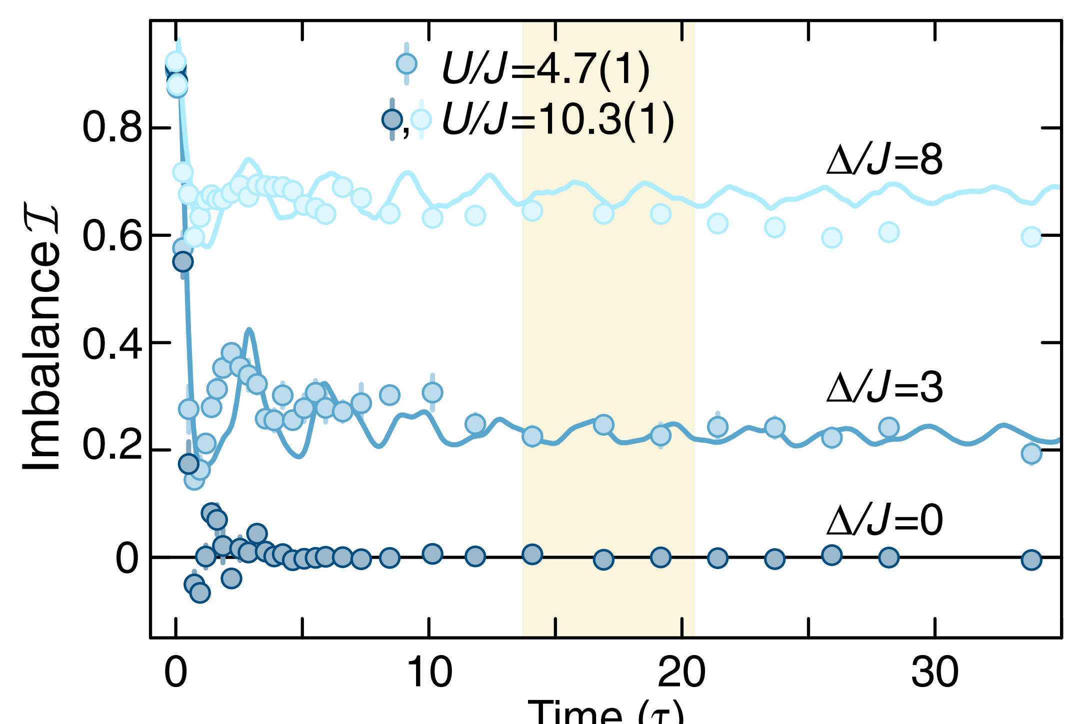
- Cold-atomic gas realization of interacting Aubry-André model:

$$\hat{H} = -J\sum_{i,\sigma}\left(\hat{c}^\dagger_{i,\sigma}\hat{c}_{i+1,\sigma} + \text{h.c.}\right) + \Delta\sum_{i,\sigma}\cos(2\pi\beta i + \phi)\hat{c}^\dagger_{i,\sigma}\hat{c}_{i,\sigma} + U\sum_i \hat{n}_{i,\uparrow}\hat{n}_{i,\downarrow}.$$

- (Non-)Equilibration of a quenched initial state measured by imbalance $\mathcal{I} = \dfrac{N_e - N_o}{N_e + N_o}$







One-dimensional trapped ions

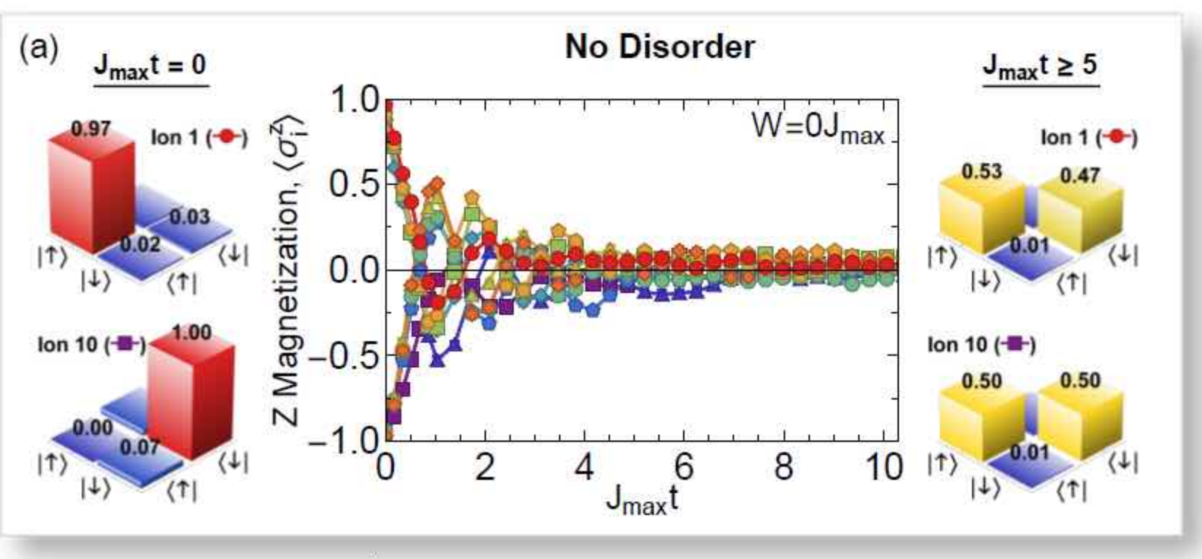
Smith *et al.*, Nat. Phys. (2016)

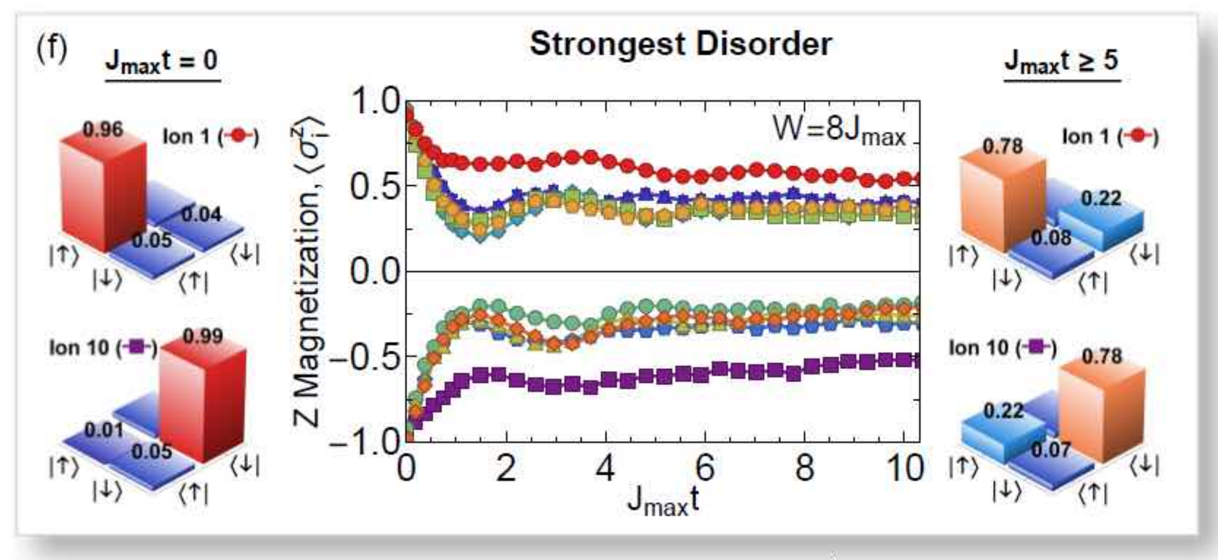
- Effective S=1/2 quantum Ising model for 10 trapped ions with ‘programmable’ interactions

$$H = \sum_{i<j} J_{ij}\sigma_i^x\sigma_j^x + B\sum_i \sigma_i^z + \sum_i D_i\sigma_i^z$$

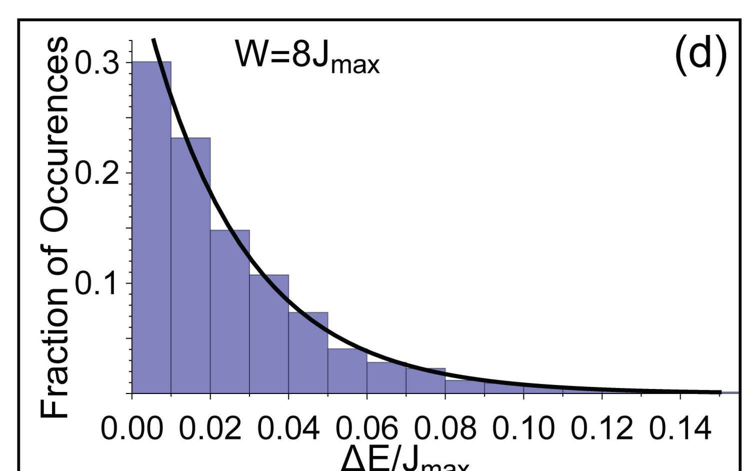
Long-range interactions $J_{ij}\ r_{ij}^{-\alpha}, \alpha \in [0.95 - 1.8]$ Randomness $D_i \in [-W, W]$

- For strong disorder, initial magnetization doesn’t decay : MBL





- Reconstructed level spacing distribution follows Poisson statistics

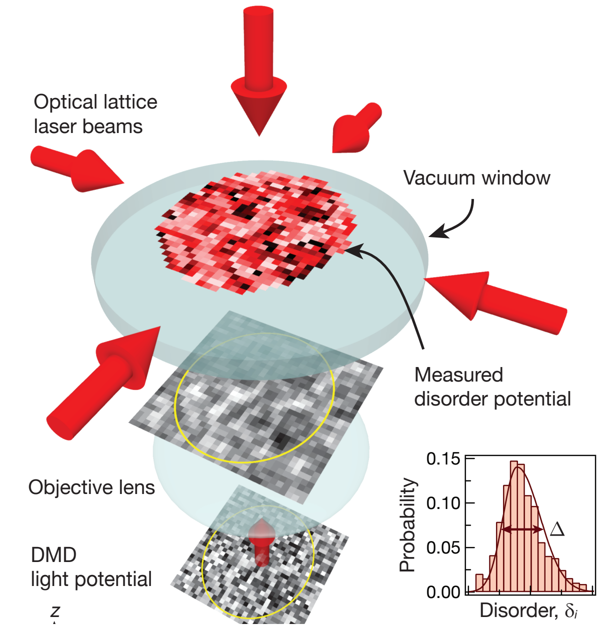


Experiments: Bosons in 2d

Choi *et al.*, Science (2016)

- Trapped cold-atomic bosonic gases with optical lattice, interactions and random disorder

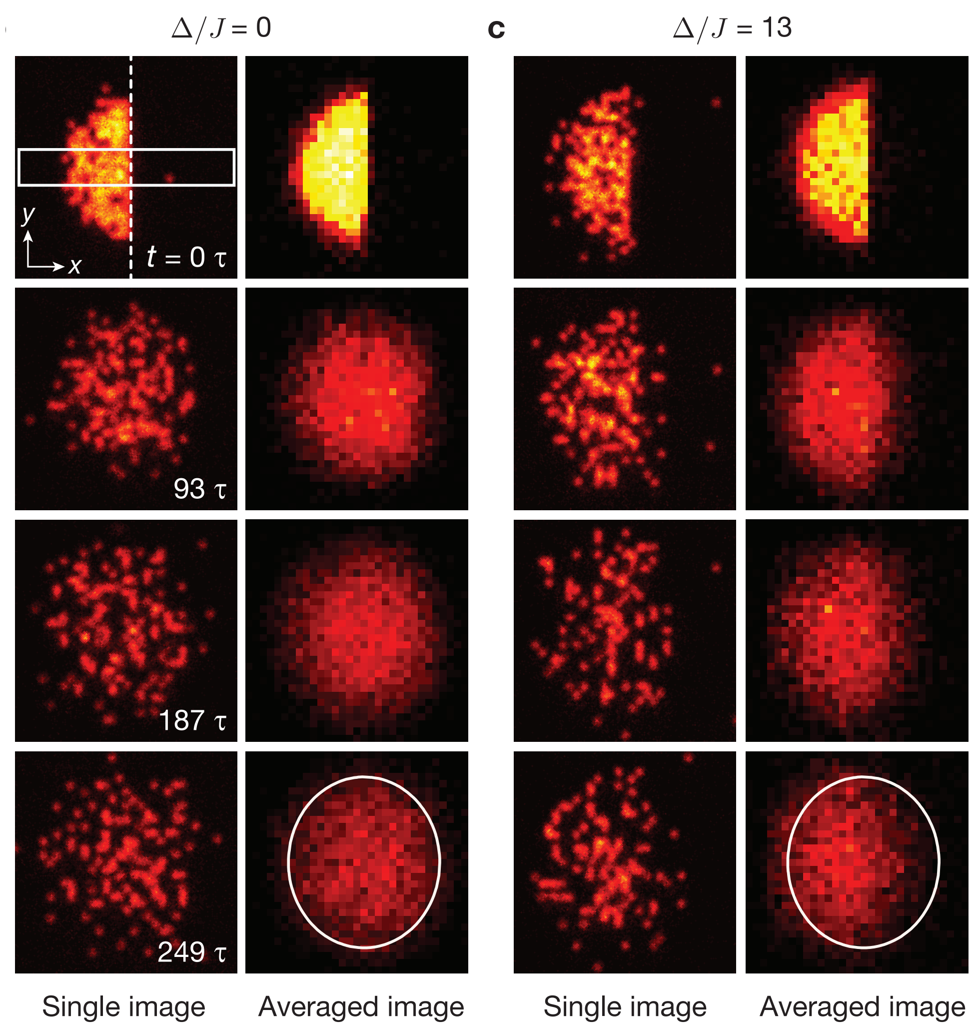
$$\hat{H} = -J \sum_{\langle \mathbf{i},\mathbf{j} \rangle} \hat{a}_{\mathbf{i}}^{\dagger} \hat{a}_{\mathbf{j}} + \frac{U}{2} \sum_{\mathbf{i}} \hat{n}_{\mathbf{i}}(\hat{n}_{\mathbf{i}} - 1) + \sum_{\mathbf{i}} (\delta_{\mathbf{i}} + V_{\mathbf{i}}) \hat{n}_{\mathbf{i}}.$$

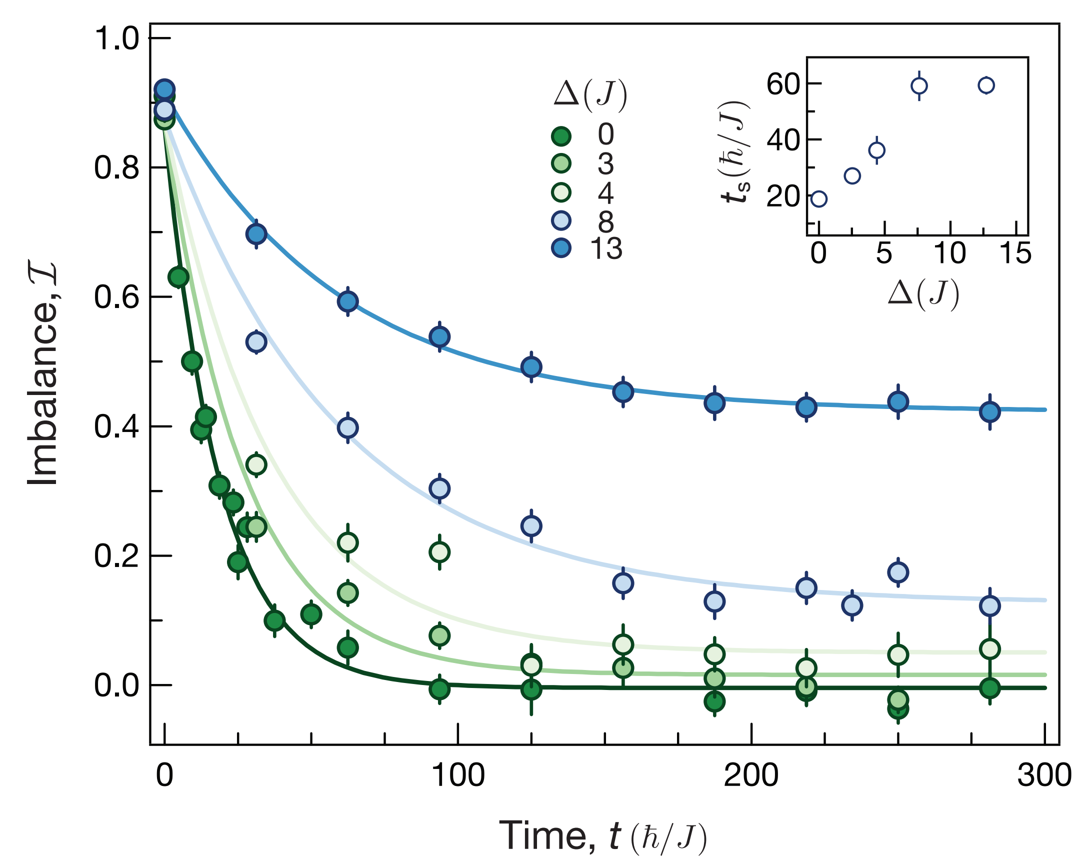


- Follow dynamics of a density domain wall

ETH

MBL



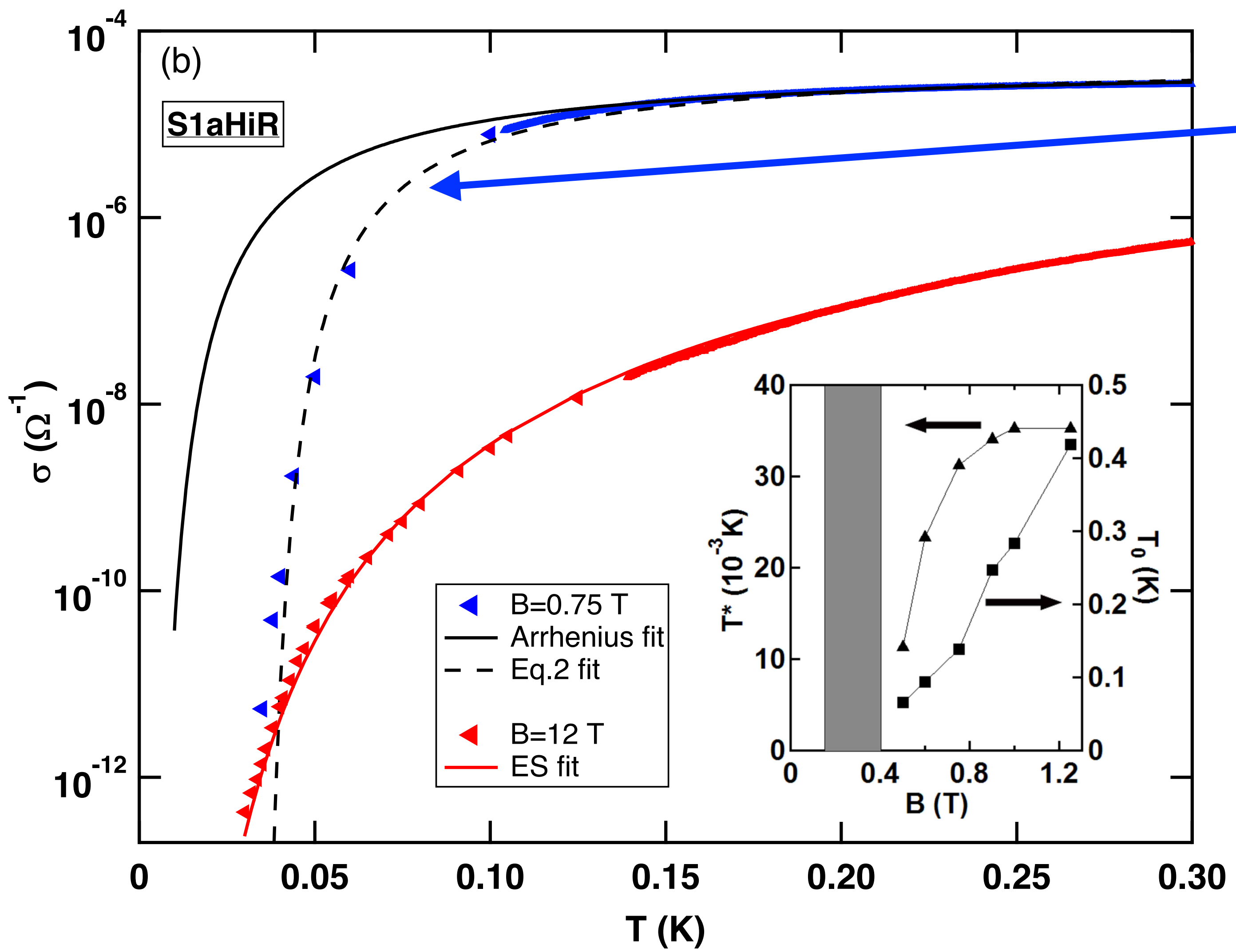


Superconducting films

Ovadia *et al.*, Sc. Rep. (2015)

- Thin films of InO can exhibit a superconducting-insulator quantum phase transition induced by a field

- Close to the critical field, at finite temperature, conductivity measurements suggest a perfect insulator at non-zero temperature : MBL ?



$$\sigma(T) = \sigma_o \exp(-T_0/(T-T_c))$$

- Existence of a finite-temperature transition implies existence of a many-body mobility edge (see later)

Behind the Scenes

MBL = Technical challenge

Technicalities

The MBL problem is very difficult to study …

- Theory : full analytical treatment of interactions + disorder impossible at finite energy density, even in 1d
- Numerics not easier : exact methods limited to small systems, standard iterative methods target extrema of spectrum, not middle! Stochastic methods (Monte Carlo) can’t be applied : MBL phase is NOT thermal
- Experiments : bath is always present!

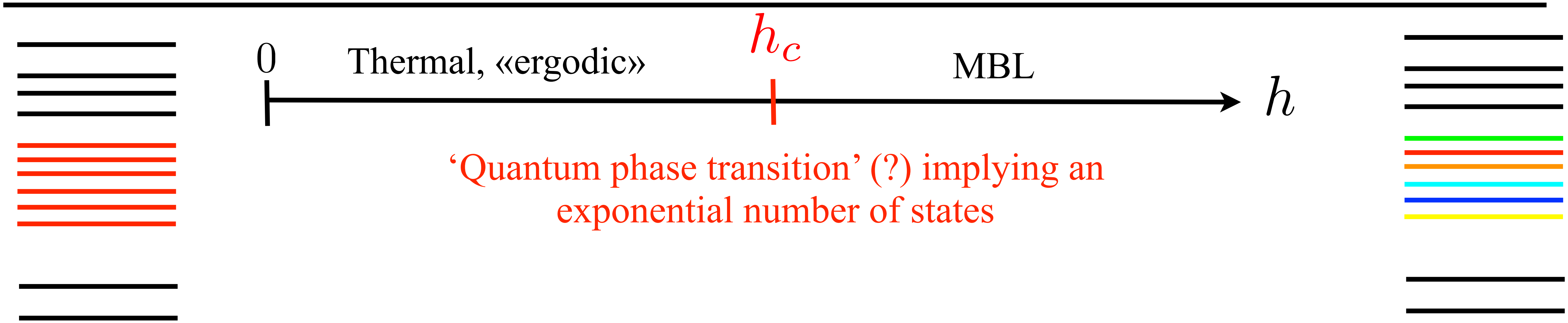
… but it expedited the creation of new methods !

- Extensions of real space renormalization group (RSRG-X) and DMRG (DMRG-X, possible because of area law) to target high-energy states
- Improved exact diagonalization schemes (shift-invert) to target middle of spectrum
- Dynamics can be followed using Time-Evolving Block Decimation techniques (because of slow growth entanglement) or in a non-equilibrium Lindblad framework using Matrix-Product States/Operators
- Race for more isolated experiments with disorder : Cold-atomic systems maybe easier to isolate and control than traditional condensed-matter ones

SEASON 4

Some new / more stuff

ETH / MBL Phase transition



- Crossover ? Phase transition : first order (?), continuous ?
- If transition, can't be seen in thermodynamics
- No clean understanding on the nature of the phase transition : Phenomenological RG suggest a new infinite disorder fixed point in 1d, with dynamical critical exponent $z = \infty$

Potter *et al.*, Vosk *et al.*

- Numerics see a transition, but exponent violate the (applicable?) Harris bound $\nu \geq 2/d$

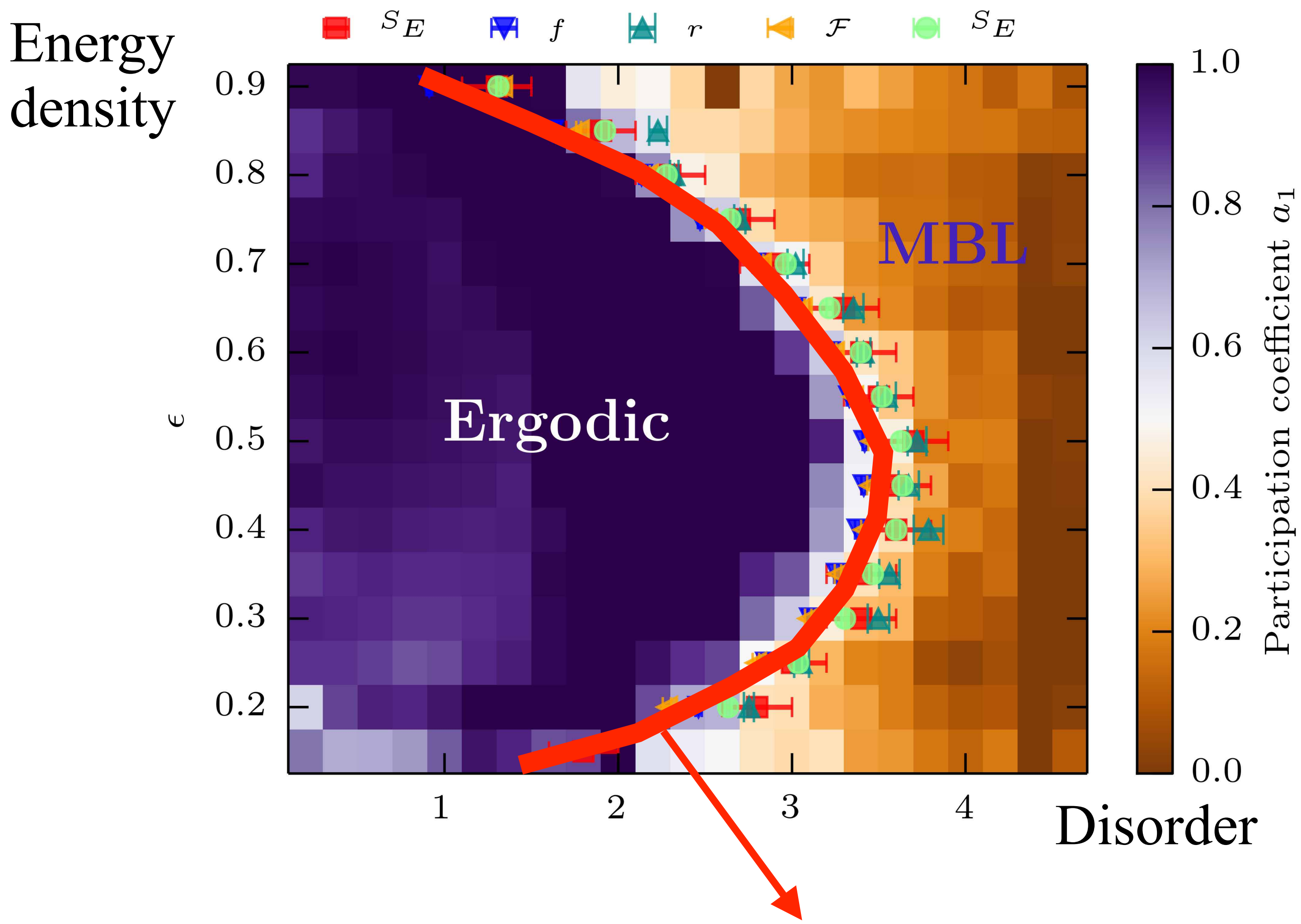
Luitz *et al.*, Kjall *et al.*

Many-body mobility edge

- First suggested by perturbative calculations : For fixed disorder, nature of states may depend on their energy. Mobility edge at extensive energy is possible

Basko, Aleiner, Altshuler

- Numerics see a mobility edge in XXZ and other models



Many body mobility edge

Numerics L=22

Luitz, Laflorencie, FA

- Existence of mobility edge debated : 'bubbles' argument

De Roeck, Huveneers, Müller

- 'Localized-bits' model NOT suited in presence of a mobility edge

MBL can host «forbidden» order

- (Discrete) quantum order can be protected by disorder

Huse *et al.*, Bauer & Nayak

Random Ising chain $H = -\sum_i h_i \sigma_i^x - \sum_i J_i \sigma_i^z \sigma_{i+1}^z$ $\mathcal{P} = \prod_i \sigma_i^x$

- No disorder, $J \gg h$: Ground-states have ferro. LRO $|0, \pm\rangle = (|\uparrow\uparrow\uparrow\uparrow\rangle_z \pm |\downarrow\downarrow\downarrow\downarrow\rangle_z)/\sqrt{2}$

Excited states are disordered (domain-walls)

MBL can host «forbidden» order

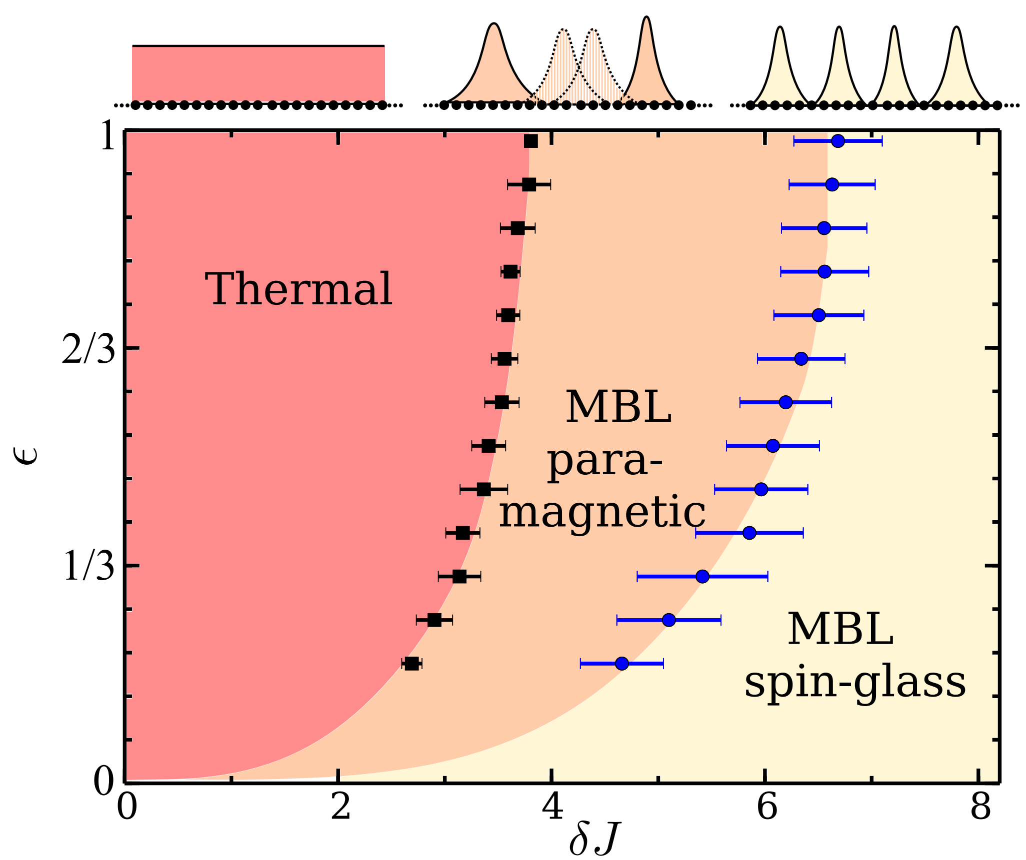
- (Discrete) quantum order can be protected by disorder

Huse *et al.*, Bauer & Nayak

Random Ising chain $H = -\sum_i h_i \sigma_i^x - \sum_i J_i \sigma_i^z \sigma_{i+1}^z$ $\mathcal{P} = \prod_i \sigma_i^x$

- Adding disorder can localize domain-walls

Excited states can now order (spin-glass fashion) $|n, \pm\rangle = (|\uparrow\uparrow\downarrow\uparrow\rangle_z \pm |\downarrow\downarrow\uparrow\downarrow\rangle_z)/\sqrt{2}$



Mermin-Wagner theorem does not apply:
Long-range order even in 1d !

Numerics in 1d model: Kjall *et al.*

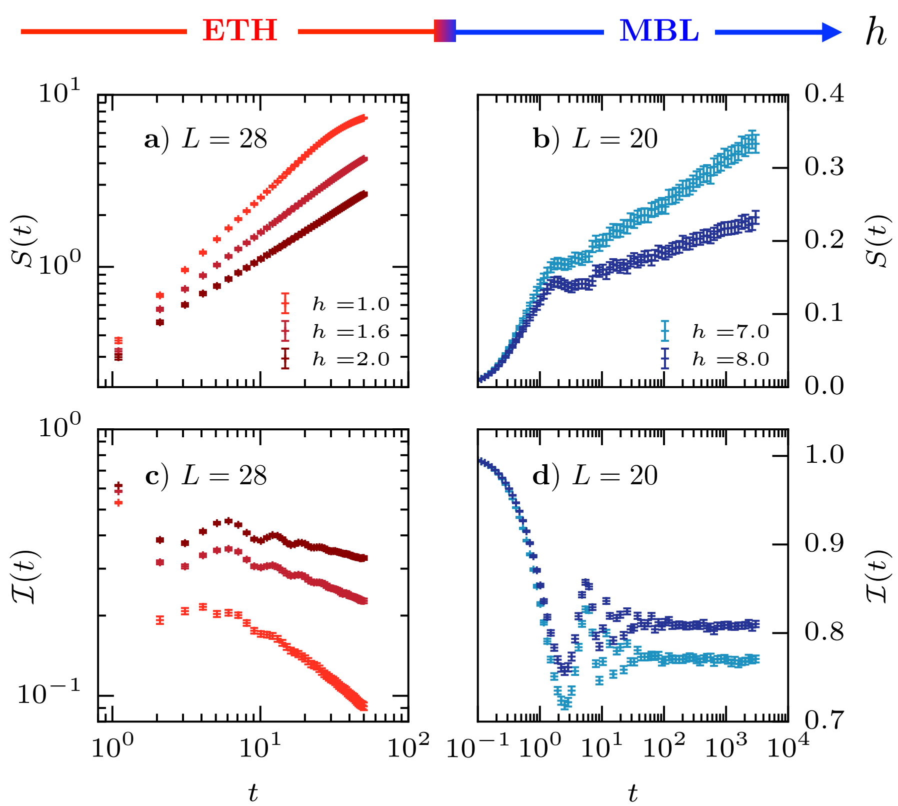
- Same argument for topological order
- MBL states could be used to build quantum memories

Precursor effect of MBL in 1d : subdiffusion

- In 1d, MBL phase can induce anomalous slowing down in the regular ETH phase (Griffiths effect)

Agarwal *et al.*

- Sub-diffusion characterized by varying power-laws even deep in the ETH phase

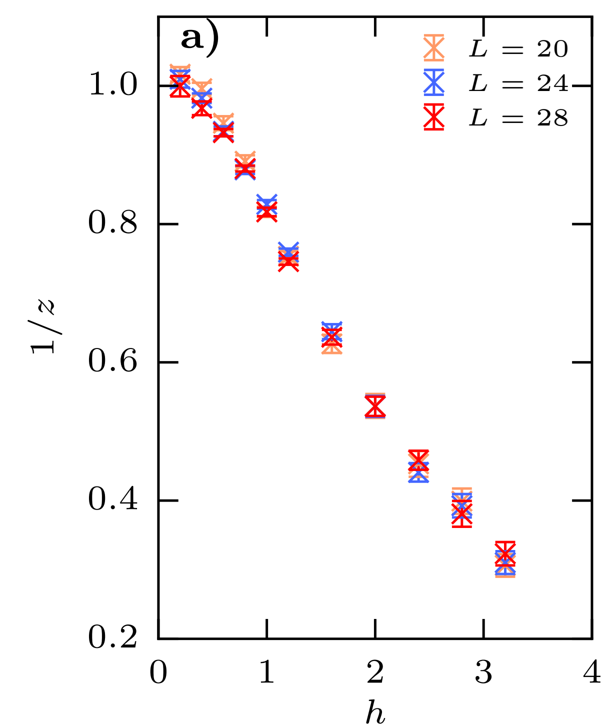


Entanglement entropy

$$S(t) \propto t^{1/z}$$

Imbalance

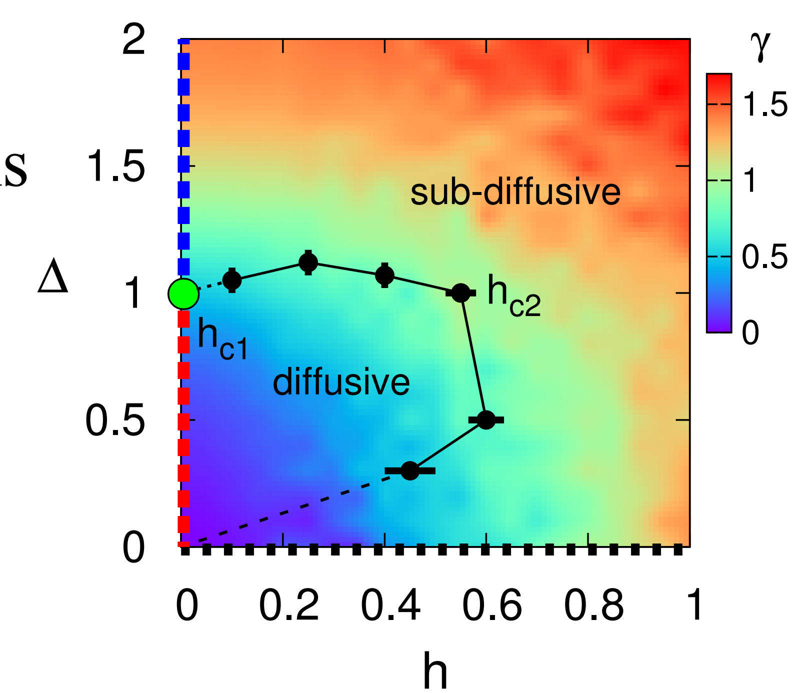
$$\mathcal{I}(t) \propto t^{-\zeta}$$



Luitz, Laflorencie, FA

- Calculation with simple local baths on large systems confirm sub-diffusion

Znidaric, Scardicchio, Varma



- Subdiffusion should NOT be there in d>1

Scenes from next episode

Open questions

Current (open) issues in the field

- Existence of mobility edge : numerics and bubble arguments in contradiction, what about local integral of motions ?
- Nature of the phase transition ? RG vs numerics ? Field theory ??

Does MBL exist in d>1 ? In the continuum ?

Can we constraint existence of MBL due to symmetry (à la Mermin-Wagner) ?

What about experiments in d=2??

Potter, Vasseur

Does MBL really need disorder ? Suggestions of translation-invariant MBL (including in Josephson Junctions). Do they pass the numerical tests?

De Roeck, Huveneers, Müller

Pino, Ioffe, Altschuler

Influence on the metallic phase : subdiffusion limited to 1d, generic?

Relation to Anderson problem (e.g. on a random graph) ?

MBL and driven / Floquet systems: avoid heating!

Quantitative description of coupling to a bath ?

Nandkishore, Gopalakrishnan

Search for MBL in other condensed-matter systems (Josephson junctions, DNP) ??

Conclusions & outlooks

- Message 1: MBL is an active interesting field! Revisits usual stat-mech, connections to different fields (many-body quantum physics, quantum chaos, quantum information, cold-atoms…)

- Message 2 : MBL is a technical challenge for theoreticians

  Field theory? Numerics are hard! New Methods?

  & experimentalists

  Realization in condensed matter / mesoscopic physics setups ?

- Message 3 : Many physics question still open!

References

- Previous reviews in Ann. Rev. Cond. Mat. Phys. (2015)

  Focus on static properties: R. Nandkishore, D. Huse, *ibid* **6**, 15 (2015) [arXiv:1404.0686]

  Dynamical properties: E. Altman & R. Vosk, *ibid* **6**, 383 (2015) [arXiv:1408.2834]

  Coupling to a bath: R. Nandkishore & S. Gopalakrishnan, arXiv:1606.08465 (intro contains review)

- Special edition of Annalen der Physik (2017) contains reviews

  Local integral of motions: J. Imbrie, V. Ros & A. Scardicchio, arXiv:1609.08076
  L. Rademaker, M. Ortuno & A. Somoza, arXiv:1610.06238

  Sub-diffusion in the ergodic side & numerics: D. Luitz & Y. Bar Lev, arXiv:1610.08993

  Transitions and dynamics: S. Parameswaran, A. Potter & R. Vasseur, arXiv:1610.03078

  Griffiths effects: K. Agarwal *et al.*, arXiv:1611.00770

  and more to come …